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On Periodic Solutions of Hamiltonian Systems of Differential Equations

T. M. Cherry

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V. *On Periodic Solutions of Hamiltonian Systems of Differential Equations.*

By T. M. CHERRY, *Ph.D., Fellow of Trinity College, Cambridge, and* 1851 *Exhibition Senior Research Student.*

(*Communicated by* Prof. H. F. BAKER, *F.R.S.*)

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CONTENTS.

INTRODUCTION.

§ 1. *Comparison with Former Theories of Periodic Solutions.*

In the following pages it is proposed to develop *ab initio* a theory of periodic solutions of Hamiltonian systems of differential equations. Such solutions are of theoretical importance for the following reason: that whereas the attempt to obtain, for a real

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Hamiltonian system, solutions valid for all real values of the independent variable t leads in general to divergent series, for certain solutions which are formally periodic the series can be proved convergent.* In the words of POINCARÉ, " ce qui nous rend ces solutions périodiques si précieuses, c'est qu'elles sont, pour ainsi dire, la seule brèche par où nous puissions essayer de pénétrer dans une place jusqu'ici réputée inabordable."†

The existing theory of periodic solutions of differential equations was developed by POINCARÉ mainly with reference to the equations of Celestial Mechanics. With a suitable choice of co-ordinates these are of the Hamiltonian form :

$$\frac{dx_k}{dt} = \frac{\partial (F_0 + \mu F_1)}{\partial y_k}, \qquad \frac{dy_k}{dt} = -\frac{\partial (F_0 + \mu F_1)}{\partial x_k} \qquad (k = 1, \dots n), \quad . \quad . \quad (1.1)‡$$

where F_0 is a function of $x_1, \dots x_n$ only and μ is a " small " coefficient of the order (mass of a planet)/(mass of the sun). It is natural therefore to take as starting point a periodic solution S of the equations when $\mu = 0$, and then, *treating* μ *as an arbitrary parameter*, to enquire whether the equations possess periodic solutions depending continuously on μ and reducing to S when $\mu = 0$. Solutions are thus found in which the co-ordinates are power series in μ, the coefficients in these series being periodic functions of t, and the series are shown to be convergent under conditions of the form

$$-\infty < t < \infty, \qquad |\mu| < \mu_0;$$

for a solution of given period T, μ_0 has a finite value, but $\mu_0 \to 0$ as $T \to \infty$.

The object of the present investigation is to study as far as possible the whole aggregate of the periodic solutions of a *definite* Hamiltonian system of equations,

$$\frac{dx_k}{dt} = \frac{\partial F}{\partial y_k}, \qquad \frac{dy_k}{dt} = -\frac{\partial F}{\partial x_k} \qquad (k = 1, \dots n), \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1.2)$$

i.e., of a system for which the Hamiltonian function F is a definite function of the co-ordinates x_k, y_k *not* involving an arbitrary parameter μ. The theory of POINCARÉ is unsuited to this object, for of the periodic solutions furnished by his theory only those whose period is not too great are valid for a *definite* non-zero value of μ fixed in advance. When we have attained some knowledge of the periodic solutions of a definite system, we can extend the theory by supposing the equations to depend on a parameter μ, and enquiring how the aggregate of periodic solutions changes as μ varies. It should be noted that even from this point of view POINCARÉ's theory is not to be considered very general, because he considers values of μ in the neighbourhood of the special value $\mu = 0$, for which the equations simplify in a far-reaching manner.

* POINCARÉ, 'Les Méthodes Nouvelles de la Mécanique Céleste.' This will be referred to as *Méth. Nouv.*

† *Op. cit.*, t. 1, p. 82.

‡ In the numbering of equations, the figure before the point refers to the section ; those following the point follow in each section the order of the natural numbers, *e.g.*, (5.9), (5.10), (5.11), etc.

Having thus defined the scope of the present investigation in comparison with POINCARÉ'S theory, we may usefully make a further comparison on one or two points.

Definition of a Periodic Solution.—A function of $f(t)$ is ordinarily said to be periodic with period T if for all t

$$f(t) = f(t + \mathrm{T}). \quad \quad (1.3)$$

It is well known that under certain conditions into which we need not enter such a function is expansible as a convergent Fourier series, or equivalently as a Laurent series in the argument $e^{2\pi\iota t/\mathrm{T}}$:

$$f(t) = \sum_{k=-\infty}^{\infty} \mathrm{A}_k e^{2\pi k\iota t/\mathrm{T}}. \quad \quad (1.4)$$

We shall have under consideration such periodic functions only as are so expansible, and shall regard the expansion (1.4) as *defining* the function: this may be called the "analytic" criterion for a periodic function in contrast with the "functional" criterion (1.3). From this point of view the quantity $2\pi\iota/\mathrm{T}$ is more fundamental than the period T; it will be called the *parameter* of the periodic function (1.4). Any solution will be called periodic in which the co-ordinates x_k, y_k are expressible as Laurent series in an argument e^{nt}; no restriction is placed on the constant n, but, of course, the period of the solution is real only if n is pure-imaginary. We call n the *parameter* of the solution.

Method by which Periodic Solutions are Investigated.—In brief, the idea is to find a contact transformation

$$x_r = f_r(z_k, u_k), \qquad y_r = g_r(z_k, u_k) \quad \quad (1.5)$$

under which the equations (1.2) become so simplified that the existence of certain particular solutions for the z_k, u_k is almost obvious. These solutions, when substituted in (1.5), furnish solutions for the x_k, y_k which turn out to be periodic in the "analytic" sense just explained. The series which specify these solutions are proved convergent by the method of Dominant Series or "calcul des limites."

POINCARÉ'S method is entirely different. He aims at finding a solution in which the co-ordinates all return to their initial values after t has increased by a certain amount. He thus uses the "functional" criterion (1.3) for a periodic function, whereas in this paper we use the "analytic" criterion (1.4).

Type of Variable and of Series Employed.—It is usual in Celestial Mechanics to employ a set of variables of which half are "angle-variables."* The series which naturally occur are then usually Fourier or multiple-Fourier series. In this paper we do not employ angle-variables, and the series which occur are always power series or Laurent

* We then regard as periodic any solution in which the angle variables return at the end of a period to their initial values increased by an integral multiple of 2π.

series.* One consequence of the employment of such series is that the question of the reality of the various solutions obtained falls somewhat into the background. To deal with this question it is necessary to start from some results concerning the reality of the fundamental transformation of § 5, but the proof of these results appears to require an uninteresting and rather detailed consideration of the manner of obtaining this transformation, which is omitted for lack of space.

For simplicity, the exposition is confined to the case of a fourth order system, *i.e.*, we suppose $n = 2$ in (1.2). There is no difficulty in extending the results to systems of higher order.

§ 2. *Ordinary and Singular Periodic Solutions.*

Before we embark on the theory of the periodic solutions of a general Hamiltonian system of the fourth order it is natural to examine the mutual relations of the periodic solutions of such special Hamiltonian systems as admit readily of complete solution. Familiar dynamical problems which lead to such "soluble" systems are the spherical pendulum, the plane motion of a particle under a central force or under two fixed centres of gravitation, etc. For definiteness we shall consider† (somewhat summarily) the system

$$\frac{dx_k}{dt} = \frac{\partial F}{\partial y_k}, \qquad \frac{dy_k}{dt} = -\frac{\partial F}{\partial x_k} \qquad (k = 1, 2), \quad \ldots \ldots \quad (2.1)$$

where

$$F = \tfrac{1}{2}\lambda_1 (x_1^2 + y_1^2) + \tfrac{1}{2}\lambda_2 (x_2^2 + y_2^2) + \tfrac{1}{2}\alpha \{y_2 (x_1^2 - y_1^2) - 2x_2 x_1 y_1\},$$

λ_1, λ_2, α being real constants which are for definiteness supposed such that $0 < \lambda_1 < \tfrac{1}{2}\lambda_2$. Put

$$x_k = r_k \sin \phi_k, \qquad y_k = r_k \cos \phi_k, \quad \ldots \ldots \quad (2.2)$$

and we find that the equations have the two integrals

$$\tfrac{1}{2}(\lambda_1 r_1^2 + \lambda_2 r_2^2) - \tfrac{1}{2}\alpha r_1^2 r_2 \cos(2\phi_1 - \phi_2) = \text{constant} = f \text{ say}, \qquad (2.3)$$

$$\tfrac{1}{2} r_1^2 + r_2^2 = \text{constant} = g^2 \text{ say}; \qquad (2.4)$$

* Those familiar with Celestial Mechanics will know how fundamentally the nature of the solution of a set of Hamiltonian equations varies according as certain "mean motions" are commensurable or incommensurable. The usual procedure employing angle-variables and the Hamilton-Jacobi method works very simply in the "incommensurable" case, but is more complicated in the "commensurable" case: one need only compare Chapters IX, XIX and XX of "Méth. Nouv." to appreciate the difference. In the present paper, by not using angle-variables we are unable conveniently to employ the Hamilton-Jacobi method, but we are able to deal, with almost equal ease, with the "incommensurable" and "commensurable" cases.

† For fuller details, see Cherry, 'Trans. Camb. Phil. Soc.,' vol. 23, p. 169 (1924), "Some examples of trajectories defined by differential equations of a generalised dynamical type," Example VII.

and that

$$\tfrac{1}{4}\left(\frac{dr_2^2}{dt}\right)^2 = \alpha^2 r_2^2 (g^2 - r_2^2)^2 - \{f - \lambda_1 g^2 - (\tfrac{1}{2}\lambda_2 - \lambda_1)\, r_2^2\}^2 \equiv \psi\,(r_2^2) \text{ say}, \quad (2.5)$$

$$\frac{d\phi_1}{dt} = \tfrac{1}{2}\lambda_2 + \frac{2f - \lambda_2 g^2}{2\,(g^2 - r_2^2)}, \qquad \frac{d\phi_2}{dt} = \tfrac{1}{2}\lambda_2 + \lambda_1 + \frac{f - \lambda_1 g^2}{r_2^2}. \quad (2.6)$$

We may thus solve by quadratures in succession for r_2, ϕ_1, ϕ_2, and r_1 is then given by (2.4). We shall consider only real solutions for the x_k, y_k, so that from (2.4) we must have $0 \leq r_2^2 \leq g^2$, and there will be a real solution only if f, g have such values that $\psi\,(r_2^2)$ is positive at one point at least of the range $0 \leq r_2^2 \leq g^2$.

We first enquire for what values of f, g the polynomial $\psi\,(r_2^2)$ has a double root in the permitted range. Suppose g to have a definite value; then when $f = \tfrac{1}{2}\lambda_2 g^2$, ψ has the double root $r_2^2 = g^2$, and there are easily shown to be two other values $f = f_1(g)$, $f = f_2(g)$, for which there are double roots $r_2^2 = a^2$, $r_2^2 = b^2$ respectively. If

$$\alpha^2 g^2 < (\tfrac{1}{2}\lambda_2 - \lambda_1)^2,$$

we have $0 < a^2 < g^2 < b^2$, and if $\alpha^2 g^2 > (\tfrac{1}{2}\lambda_2 - \lambda_1)^2$ we have $0 < a^2 < b^2 < g^2$. When, say, $f = f_1(g)$, we have for r_2^2 the "equilibrium-solution" $r_2^2 = a^2$, so from (2.3), (2.4), (2.6), r_1^2 and $(2\phi_1 - \phi_2)$ also are constant, and ϕ_1, ϕ_2 are linear functions of t; these solutions, when substituted in (2.2), give *periodic* solutions for the x_k, y_k. We thus find explicitly the following periodic solutions:

Family I:

$$x_1 = y_1 = 0,\ \ x_2 = g \sin \lambda_2\,(t + \varepsilon),\ \ y_2 = g \cos \lambda_2\,(t + \varepsilon) \quad g, \varepsilon \text{ arbitrary constants}; \quad (2.7)$$

Families, II, III:

$$\left.\begin{aligned} \alpha x_1 &= \{2\,(\lambda_1 - \mu)\,(\lambda_2 - 2\mu)\}^{\frac{1}{2}} \sin \mu\,(t + \varepsilon), & \alpha x_2 &= (\lambda_1 - \mu) \sin 2\mu\,(t + \varepsilon) \\ \alpha y_1 &= \{2\,(\lambda_1 - \mu)\,(\lambda_2 - 2\mu)\}^{\frac{1}{2}} \cos \mu\,(t + \varepsilon), & \alpha y_2 &= (\lambda_1 - \mu) \cos 2\mu\,(t + \varepsilon) \end{aligned}\right\}, \quad (2.8)$$

where ε is an arbitrary constant* and μ is a root (the smaller root for Family II, the larger for Family III) of the equation

$$(\lambda_1 - \mu)\,(\lambda_1 + \lambda_2 - 3\mu) = \alpha^2 g^2.$$

Family II is real for $g^2 \geq 0$ and has $\mu \leq \lambda_1$, while Family III is real for $\alpha^2 g^2 \geq (\tfrac{1}{2}\lambda_2 - \lambda_1)^2$, and has $\mu \geq \tfrac{1}{2}\lambda_2$.

When f, g have such values that $\psi(r_2^2)$ has two simple roots $r_2^2 = \beta^2$, γ^2 such that $0 < \beta^2 < \gamma^2 < g^2$ and also that $\psi > 0$ when $\beta^2 < r_2^2 < \gamma^2$, (2.5) gives for r_2^2 a real periodic solution having the period

$$\mathrm{T} = \int_{\beta^2}^{\gamma^2} dr_2^2/\{\psi\,(r_2^2)\}^{\frac{1}{2}}. \quad (2.9)$$

* Here and below (in this section) any ε is understood to be an arbitrary constant of integration.

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For our purpose we shall consider only values of g^2 in the range $0 < g^2 < (\frac{1}{2}\lambda_2 - \lambda_1)^2/\alpha^2$, and values of f in the neighbourhood of $f = \frac{1}{2}\lambda_2 g^2$. We find that when $f < \frac{1}{2}\lambda_2 g^2$ there are roots $r_2^2 = \beta^2$, γ^2 satisfying these conditions, so the equations (2.6) become of the form

$$\frac{d\phi_1}{dt} = C_1 + \sum_{m=1}^{\infty} \{C'_m \cos 2\pi m (t+\varepsilon)/T + D'_m \sin 2\pi m (t+\varepsilon)/T\},$$

$$\frac{d\phi_2}{dt} = C_2 + \sum_{m=1}^{\infty} \{C''_m \cos 2\pi m (t+\varepsilon)/T + D''_m \sin 2\pi m (t+\varepsilon)/T\},$$

where the C, D are constants depending on f, g. The solution is of the form

$$\phi_1 - C_1 (t + \varepsilon_1) = \text{periodic function of } (t+\varepsilon) \text{ (period T)},$$

$$\phi_2 - C_2 (t + \varepsilon_2) = \text{periodic function of } (t+\varepsilon) \text{ (period T)}.$$

Now the existence of the relation (2.3) shows that the arguments $C_1 (t+\varepsilon_1)$, $C_2 (t+\varepsilon_2)$, $2\pi (t+\varepsilon)/T$ cannot be independent; in fact, $\cos (2\phi_1 - \phi_2)$ is periodic with period T, and $(\phi_2 - 2\phi_1)$ increases by 2π when t increases by T,* so that we must have

$$C_2 - 2C_1 = 2\pi/T. \quad \ldots \ldots \ldots \ldots \quad (2.10)$$

Thus the x_k, y_k are doubly-periodic functions (period 2π) of the two arguments

$$C_1(t+\varepsilon_1), \qquad C_2(t+\varepsilon_2),$$

say,

$$\left. \begin{aligned} x_k &= \theta_k \{C_1 (t+\varepsilon_1), \quad C_2 (t+\varepsilon_2), \quad f, g^2\} \\ y_k &= \chi_k \{C_1 (t+\varepsilon_1), \quad C_2 (t+\varepsilon_2), \quad f, g^2\} \end{aligned} \right\}, \quad \ldots \ldots \quad (2.11)$$

and it is easily seen that C_1, C_2 are continuous functions of f, g, and that θ_k, χ_k are continuous functions of their four arguments. We thus have in (2.11) the general solution of the equations (2.1), valid for values of f, g in the range

$$0 < g^2 < (\tfrac{1}{2}\lambda_2 - \lambda_1)^2/\alpha^2, \qquad \lambda_1 g^2 < f < \tfrac{1}{2}\lambda_2 g^2.$$

It may be shown that

$$C_1 = \tfrac{1}{2}\lambda_2 + \frac{(2f - \lambda_2 g^2)}{2T} \int_{\beta^2}^{\gamma^2} \frac{dr_2^2}{(g^2 - r_2^2)\{\psi(r_2^2)\}^{\frac{1}{2}}},$$

$$C_2 = \tfrac{1}{2}\lambda_2 + \lambda_1 + \frac{(f - \lambda_1 g^2)}{T} \int_{\beta^2}^{\gamma^2} \frac{dr_2^2}{r_2^2 \{\psi(r_2^2)\}^{\frac{1}{2}}}, \quad \ldots \quad (2.12)$$

and thence the relation (2.10) may be verified. We may show further that as $f \to \frac{1}{2}\lambda_2 g^2$, so that the roots $r_2^2 = \beta^2$, γ^2 tend to coincidence at $r_2^2 = g^2$, C_1 and C_2 tend continuously to the values:

$$C_1 \to \tfrac{1}{2}\lambda_2 - \{(\tfrac{1}{2}\lambda_2 - \lambda_1)^2 - \alpha^2 g^2\}^{\frac{1}{2}}, \qquad C_2 \to \lambda_2, \quad \ldots \ldots \quad (2.13)$$

* This is true only when f lies between $\frac{1}{2}\lambda_2 g^2$ and $\lambda_1 g^2$; when $f < \lambda_1 g^2$, $\phi_2 - 2\phi_1$ returns to its initial value when t increases by T (see the author's paper cited above, footnote to p. 198).

and the solution (2.11) changes continuously into

$$x_1 = y_1 = 0, \qquad x_2 = g \sin \lambda_2 (t + \varepsilon_2), \qquad y_2 = g \cos \lambda_2 (t + \varepsilon_2),$$

viz., into the Family I of periodic solutions.

We are now in a position to investigate all the periodic solutions which exist for values of f, g in the range under consideration. The solution obtained from (2.11) by giving f, g, ε_1, ε_2 any definite values is periodic provided C_1 and C_2 are commensurable; in fact, if f, g have such values that

$$C_1 = 2n_1\pi/T_0, \qquad C_2 = 2n_2\pi/T_0, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2.14)$$

where n_1, n_2 are integers mutually prime, (2.11) gives a single infinity* of periodic solutions all having the period T_0. Moreover, the relation

$$C_1/C_2 = n_1/n_2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2.15)$$

is a continuous relation connecting f and g, so we may leave g arbitrary and determine f in terms of g so that (2.14) are satisfied, T_0 being now a continuous function of g. Thus, with g arbitrary and f so determined, (2.11) gives a continuous *doubly-infinite* family of periodic solutions, the period T_0 varying continuously from member to member of this family, and being, of course, constant over singly-infinite sub-families; there is a family of this sort corresponding to every commensurable ratio n_1/n_2, the period T_0 being the lowest common integral multiple of $2\pi/C_1$, $2\pi/C_2$.

Now for suitable values of the ratio n_1/n_2 there will be values of f, g which satisfy simultaneously the relation (2.15) and

$$f = \tfrac{1}{2}\lambda_2 g^2, \qquad 0 < \alpha^2 g^2 < (\tfrac{1}{2}\lambda_2 - \lambda_1)^2; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2.16)$$

for we have seen that as $f \to \tfrac{1}{2}\lambda_2 g^2$, $C_1/C_2 \to \tfrac{1}{2} - \{(\tfrac{1}{2}\lambda_2 - \lambda_1)^2 - \alpha^2 g^2\}^{\frac{1}{2}}/\lambda_2$, so that for any value of n_1/n_2 between λ_1/λ_2 and $\tfrac{1}{2}$ there are values of f, g satisfying (2.15), (2.16). Hence to the doubly-infinite family of periodic solutions corresponding to such a commensurable value of n_1/n_2 belongs as a limiting member a periodic solution of Family I, viz., that solution for which $\tfrac{1}{2} - \{(\tfrac{1}{2}\lambda_2 - \lambda_1)^2 - \alpha^2 g^2\}^{\frac{1}{2}}/\lambda_2 = n_1/n_2$. This solution is described as a limiting member of the family because—

(i) Its period is $2\pi/\lambda_2$, whereas for neighbouring solutions of the family the period T_0, $= 2n_2\pi/C_2$ tends as $f \to \tfrac{1}{2}\lambda_2 g^2$ to the value $2n_2\pi/\lambda_2$, and n_2 is an integer in general greater than 1.

(ii) When $f \neq \tfrac{1}{2}\lambda_2 g^2$ there is a single infinity of solutions having the same period T_0, whereas there is only one limiting solution having the period $2\pi/\lambda_2$.

(iii) The doubly-infinite family is real for $f < \tfrac{1}{2}\lambda_2 g^2$, but unreal for $f > \tfrac{1}{2}\lambda_2 g^2$.

* We do not regard as distinct two solutions of which one differs from the other only by the addition of a constant to t, so of the constants ε_1, ε_2 which remain arbitrary one may be suppressed.

We may regard this doubly-infinite family as *branching* from Family I at the solution for which $\frac{1}{2} - \{(\frac{1}{2}\lambda_2 - \lambda_1)^2 - \alpha^2 g^2\}^{\frac{1}{2}}/\lambda_2 = n_1/n_2$, and there is such a family branching from each solution of Family I for which $\frac{1}{2} - \{(\frac{1}{2}\lambda_2 - \lambda_1)^2 - \alpha^2 g^2\}^{\frac{1}{2}}/\lambda_2$ is a commensurable number between λ_1/λ_2 and $\frac{1}{2}$. It is readily shown that this quantity is the ratio of one of the non-zero exponents* of the solution (2.7) to its parameter $\iota\lambda_2$, and for any periodic solution we call the corresponding quantity its *characteristic ratio*.

We now enquire whether there is a corresponding doubly-infinite family of periodic solutions for which $n_1/n_2 = \frac{1}{2}$. If this is so we see from (2.10), (2.15) that T will be infinite, so the values of f, g must be such that $\psi(r_2^2)$ has a double root; on this account the family in question, if it exists, must be furnished by some limiting form of the general solution (2.11). Now for Family III, given by (2.8) with $\mu \geq \frac{1}{2}\lambda_2$, we have obviously $C_1/C_2 = \frac{1}{2}$, and it has as limiting member when $\mu = \frac{1}{2}\lambda_2$, $\alpha^2 g^2 = (\frac{1}{2}\lambda_2 - \lambda_1)^2$ the solution

$$x_1 = y_1 = 0, \quad \alpha x_2 = (\lambda_1 - \tfrac{1}{2}\lambda_2) \sin \lambda_2 (t + \varepsilon), \quad \alpha y_2 = (\lambda_1 - \tfrac{1}{2}\lambda_2) \cos \lambda_2 (t + \varepsilon),$$

which is the solution of Family I whose characteristic ratio is $\frac{1}{2}$. We see, then, that from this solution there *does* branch a family of periodic solutions, but this family (viz., Family III) is singly- and not doubly-infinite. It resembles the previous doubly-infinite families in that as $f \to \frac{1}{2}\lambda_2 g^2$ the period $2\pi/\mu$ tends to the value $2n_2\pi/\lambda_2$, n_2 having the value 2.

By a similar method we can investigate the periodic solutions which exist for values of f, g outside the range we have been considering. It will be found that for the periodic solutions of Families II, III, the characteristic ratio (viz., the ratio of a non-zero exponent to the parameter $\iota\mu$) is real, and that doubly-infinite families of periodic solutions branch from those solutions for which this ratio is rational. On the other hand, for those solutions of Family I for which $\alpha^2 g^2 > (\frac{1}{2}\lambda_2 - \lambda_1)^2$ the characteristic ratio is unreal, and there are no families branching from this portion of Family I.

It appears to be true that for any fourth order Hamiltonian system of which the complete solution can be carried through, the families of periodic solutions are of a similar nature to those just considered in detail.† We may summarise as follows:—

The periodic solutions are of two kinds, which we shall call *ordinary* and *singular*, respectively.‡ Ordinary periodic solutions occur in continuous doubly-infinite families over which the period varies continuously, being constant over singly-infinite sub-families; all their exponents are zero,§ so they have their characteristic ratio zero. Singular periodic solutions occur in continuous singly-infinite families over which the period varies continuously; their characteristic ratio is in general non-zero, and varies continuously over the family. Let $\mathfrak{S}$ be a singular periodic solution belonging to a family

* For definition, see § 4 below.

† For some examples of soluble systems, see the author's paper cited above. On this classification of periodic solutions, compare WHITTAKER, 'Proc. Roy. Soc. Edin.' (1916), "On the adelphic integral of the differential equations of dynamics."

‡ WHITTAKER, *loc. cit.*, § 1.

§ *Cf.* "Méth. Nouv.," No. 148.

$\mathfrak{F}$, the characteristic ratio of $\mathfrak{S}$ being real and rational; then, in general, there branches from the family $\mathfrak{F}$ at $\mathfrak{S}$ a family of ordinary periodic solutions, for which the period tends continuously in the neighbourhood of $\mathfrak{S}$ to an integral multiple of the period of $\mathfrak{S}$. Exceptionally, however—*i.e.*, at isolated members of the family $\mathfrak{F}$—there may branch a finite number of singly-infinite families of singular periodic solutions instead of one doubly-infinite family of ordinary periodic solutions.

The most striking result of the investigation which follows is that *for fourth order Hamiltonian systems in general, the periodic solutions are in general "singular"*: they satisfy in general the above definition in that they occur in singly-infinite families and have their characteristic ratio in general non-zero. We show, in fact, that any periodic solution $\mathfrak{S}$ belongs in general to a *singly-infinite* family $\mathfrak{F}$, along which the period and the characteristic ratio vary continuously, and that if $\mathfrak{S}$ has its characteristic ratio real and rational, there branch in general from $\mathfrak{F}$ at $\mathfrak{S}$ a finite number of singly-infinite families (in general two) of periodic solutions, along which also the period and the characteristic ratio vary continuously.

This result indicates that Hamiltonian systems of equations may be divided into two categories according to the nature of their periodic solutions. For systems of the first category the periodic solutions are in general "ordinary": such systems may be appropriately described as *soluble* since apparently all readily soluble systems are here included. For systems of the second category the periodic solutions are in general "singular," and they may be described as *insoluble*, since here apparently all systems which have hitherto resisted solution are included.*

Preliminary Definitions and Results.

§ 3. *Lagrange Brackets and Contact Transformations.*

If $2n$ variables $x_1, \ldots x_n, y_1, \ldots y_n$ are functions of any number of other variables $z_1, z_2, \ldots,$ the expression

$$\sum_{k=1}^{n}\left(\frac{\partial x_k}{\partial z_r}\frac{\partial y_k}{\partial z_s}-\frac{\partial x_k}{\partial z_s}\frac{\partial y_k}{\partial z_r}\right)$$

is called the *Lagrange bracket* of z_r and z_s and is written

$$[z_r, z_s] \quad \text{or} \quad [z_r, z_s]_{xy},$$

the latter notation being used when there is risk of confusion through other functions of the z_k beside the x_k and y_k being under consideration. It is easy to prove the identity

$$\frac{\partial}{\partial z_p}[z_r, z_s]+\frac{\partial}{\partial z_r}[z_s, z_p]+\frac{\partial}{\partial z_s}[z_p, z_r]=0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3.1)$$

* A similar designation of dynamical equations as "integrable" or "non-integrable" will be found in Birkhoff, 'Rend. Circ. Mat. Palermo,' t. 39, p. 1 (1915); 'Trans. Amer. Math. Soc.,' vol. 18, p. 199 (1917); and in other places.

The conditions that a transformation

$$\left.\begin{array}{l} x_k = \phi_k(z_1, \dots z_n, u_1, \dots u_n, t) \\ y_k = \psi_k(z_1, \dots z_n, u_1, \dots u_n, t) \end{array}\right\} \quad (k = 1, \dots n),$$

or, as we shall write for short—

$$x_k = \phi_k(z_r, u_r, t), \qquad y_k = \psi_k(z_r, u_r, t), \quad \dots \dots \dots \quad (3.2)$$

may be a contact transformation, *i.e.*, shall transform any Hamiltonian system of differential equations into another Hamiltonian system, may be expressed as follows: Taking the Hamiltonian system

$$\frac{dx_k}{dt} = \frac{\partial F}{\partial y_k}, \qquad \frac{dy_k}{dt} = -\frac{\partial F}{\partial x_k} \qquad (k = 1, \dots n),$$

suppose that on expressing F in terms of the z_k, u_k, t by means of (3.2) we obtain

$$F(x_k, y_k, t) \equiv H(z_k, u_k, t).$$

Then we have

$$\left.\begin{array}{l} \dfrac{\partial H}{\partial z_r} = [t, z_r] + \dfrac{dz_1}{dt}[z_1, z_r] + \dots + \dfrac{du_n}{dt}[u_n, z_r] \\ \dfrac{\partial H}{\partial u_r} = [t, u_r] + \dfrac{dz_1}{dt}[z_1, u_r] + \dots + \dfrac{du_n}{dt}[u_n, u_r] \end{array}\right\} \quad \dots \dots \dots \quad (3.3)$$

Now suppose the conditions

$$\left.\begin{array}{ll} [z_r, z_s] = [u_r, u_s] = 0 & (r, s = 1, \dots n) \\ [z_r, u_s] = 0 & (r, s = 1, \dots n;\ r \neq s) \\ [z_r, u_r] = 1 & (r = 1, \dots n) \end{array}\right\} \quad \dots \dots \dots \quad (3.4)$$

are satisfied; then from the identity (3.1) it is easily proved that

$$\frac{\partial}{\partial \alpha}[t, \beta] = \frac{\partial}{\partial \beta}[t, \alpha],$$

where α, β stand for any two of the z_k, u_k, so that there exists a function M of the z_k, u_k, t such that

$$[t, z_r] = -\frac{\partial M}{\partial z_r}, \qquad [t, u_r] = -\frac{\partial M}{\partial u_r}. \quad \dots \dots \dots \quad (3.5)$$

Thus, in virtue of (3.4), (3.5), the equations (3.3) reduce to

$$\frac{dz_r}{dt} = \frac{\partial K}{\partial u_r}, \qquad \frac{du_r}{dt} = -\frac{\partial K}{\partial z_r} \ (r = 1, \dots n);$$

the new Hamiltonian function K is equal to F + M, where M depends only on the transformation (3.2) and is defined by the consistent equations (3.5).

§ 4. *The Exponents of a Periodic Solution.*

Let

$$\frac{dx_k}{dt} = X_k(x_1, \ldots x_n) \quad (k = 1, \ldots n) \quad \ldots \ldots \ldots \quad (4.1)$$

be a set of differential equations having the periodic solution

$$x_k = \phi_k(t) \quad (k = 1, \ldots n) \quad (4.2)$$

of period T. Neighbouring solutions of (4.1) are obtained by making the transformation

$$x_k = \phi_k(t) + \xi_k \qquad (k = 1, \ldots n) \quad (4.3)$$

and solving the resulting equations for the ξ_k. If powers of these variables higher than the first are neglected, we obtain a set of homogeneous linear equations with periodic coefficients, called the *variational equations*,* formed from (4.2) as *generating solution*. The form of the solution of such equations is well known† to be

$$\xi_k = c_1 e^{\lambda_1 t}\, \psi_{k1}(t) + \ldots + c_n e^{\lambda_n t}\, \psi_{kn}(t), \quad (4.4)$$

where the ψ_{rs} are periodic functions of t, the λ_k are definite constants called the *characteristic exponents*,‡ or, for short, the *exponents* of the solution (4.2), and the c_k are the arbitrary constants of integration. As defined by (4.4), the exponents are indeterminate to an arbitrary integral multiple of $2\pi\iota/T$: we may make the definition precise by requiring that for each exponent λ_k the real part R_k of the ratio $\lambda_k T/2\pi\iota$ shall satisfy the condition $-\frac{1}{2} < R_k \leq \frac{1}{2}$.

When two or more of the exponents are equal the form (4.4) of the solution of the variational equations must be modified in general by the introduction of terms polynomial in t.

It may be remarked that the validity of these results is in no way dependent upon T being real, but this is, of course, the important case from the dynamical point of view.

When the system (4.1) has the Hamiltonian form the exponents fall into pairs, the sum of those in any pair being zero.§ If, moreover, the functions $\phi_k(t)$ in (4.2) are not all constants, two of the exponents are always zero.

Suppose the fourth order Hamiltonian system

$$\frac{dx_k}{dt} = \frac{\partial F}{\partial y_k}, \qquad \frac{dy_k}{dt} = -\frac{\partial F}{\partial x_k} \qquad (k = 1, 2), \quad \ldots \ldots \ldots \quad (4.5)$$

in which F is an analytic function of x_1, y_1, x_2, y_2, possesses the periodic solution

$$x_k = \phi_k(t), \qquad y_k = \psi_k(t) \qquad (k = 1, 2). \quad \ldots \ldots \ldots \quad (4.6)$$

* "Méth. Nouv.," Chap. IV.

† See, for example, "Méth. Nouv.,' No. 29.

‡ "Méth. Nouv.," Chap. IV.

§ "Méth. Nouv.," No. 69.

For the solutions which will be under consideration, ϕ_k, ψ_k will be developable as convergent Laurent series in an argument $e^{\nu t}$, and the constant ν will be called the *parameter* of the solution; it is connected with the period T by the relation $\nu T = 2\pi\iota$. If the ϕ_k, ψ_k are not all constants two of the exponents of the solution (4.6) are zero, and the other two, say $\pm\lambda$, which are in general non-zero, have their sum zero. The quantity $\lambda/\nu\iota$ will be called the *characteristic ratio* for the solution (4.6); its real part R satisfies $0 \leq R \leq \frac{1}{2}$.

If the functions ϕ_k, ψ_k are all constants we call (4.6) an *equilibrium solution* of the equations (4.5). It may be considered, of course, as a special case of a periodic solution. Of its four exponents $\pm\lambda_1$, $\pm\lambda_2$, none is in general zero. In the variational equations

$$\frac{d\xi_k}{dt} = a_{k1}\xi_1 + \ldots + a_{k4}\xi_4 \quad (k = 1, \ldots 4),$$

the a_{rs} are now constants, and the exponents are the roots of the equation

$$\begin{vmatrix} a_{11} - \lambda, & a_{12}, & \ldots \\ a_{21}, & a_{22} - \lambda, & \ldots \\ \ldots, & \ldots, & \ldots \end{vmatrix} = 0.$$

In the investigation which follows we start from the hypothesis that for the system (4.5) we know *one* periodic solution $\mathfrak{H}$, and the bulk of the paper is occupied with finding, *for the neighbourhood of* $\mathfrak{H}$, those families of periodic solutions to which $\mathfrak{H}$ belongs. Any solution thus found can be made the base for a similar investigation, and so on. In the last section an attempt is made to synthesise the knowledge gained from such "local" investigations, so as to gain some idea of the nature of the totality of the periodic solutions possessed by the equations. The initial hypothesis that the equations possess one periodic solution will be seen later to be of a not very restrictive character. For general investigations relating to this point reference may be made to BIRKHOFF, 'Trans. Amer. Math. Soc.,' vol. 18, p. 199 (1917), "Dynamical Systems with Two Degrees of Freedom," and to WHITTAKER, "Analytical Dynamics" (3rd edition), § 168.

The work of §§ 5–11 falls into two parts:

(i) If we make a transformation to new variables z_k, u_k:

$$x_k = f_k(z_r, u_r, t), \qquad y_k = g_k(z_r, u_r, t),$$

in which the functions f_k, g_k are as regards t periodic with parameter σ, any periodic solution for the z_k, u_k of parameter σ will transform into a periodic solution for the x_k, y_k. We find a transformation which so simplifies the equations (4.5) that the existence of periodic solutions is almost obvious. The transformation gives the old variables x_k, y_k as power series in the new variables z_k, u_k, but as these series cannot be proved convergent this theory proves only the *formal* existence of periodic solutions for the x_k, y_k.

(ii) When the formal existence of periodic solutions has been proved (§§ 5–9), it is easy to give a process for their direct construction, and the resulting series may be proved convergent by the method of dominant functions (§§ 10, 11).

The details of the theory differ according as the known solution $\mathfrak{S}$, from which we start, is or is not an equilibrium solution, though the main lines are the same in the two cases. We shall treat in detail the general case only in which the periodic solution $\mathfrak{S}$ is not an equilibrium solution.

Periodic Solutions in the Neighbourhood of a Known Periodic Solution.

§ 5. *Transformation of the Equations.*

We consider fourth order Hamiltonian systems of differential equations

$$\frac{dx_k}{dt}=\frac{\partial \mathrm{F}}{\partial y_k},\qquad \frac{dy_k}{dt}=-\frac{\partial \mathrm{F}}{\partial x_k}\qquad (k=1,2), \quad \dots \dots \dots \quad (5.1)$$

in which the Hamiltonian function F is an analytic function of the x_k, y_k not involving t. For these equations we suppose that we know a periodic solution of period T :

$$x_k=\phi_k(t),\qquad y_k=\psi_k(t)\qquad (k=1,2). \quad \dots \dots \dots \quad (5.2)$$

We suppose that F is developable about every point of this solution, and is single-valued in the neighbourhood of the solution.* Then, if in (5.1) we change the variables by the contact transformation

$$x_k=\phi_k(t)+\xi_k,\qquad y_k=\psi_k(t)+\eta_k \quad \dots \dots \dots \quad (5.3)$$

we obtain for the ξ_k, η_k equations of the form

$$\frac{d\xi_k}{dt}=\Xi_k,\qquad \frac{d\eta_k}{dt}=\mathrm{H}_k, \quad \dots \dots \dots \quad (5.4)$$

in which the Ξ_k, H_k are power series in the ξ_k, η_k with coefficients periodic in t.† Since (5.2) is a solution of (5.1), the Ξ_k, H_k vanish with the ξ_k, η_k.

We shall treat here the general case in which (5.2) is not an equilibrium solution; for the special case of an equilibrium solution, see § 12 below.

First Transformation.—If on the right of (5.4) we neglect terms of higher order than 1

* The meaning of this phrase is as follows: Let t start from a value t_0 and change continuously to a value t_1, $=t_0+n\mathrm{T}$ (n integral); then (5.2) define, in the (complex) planes of the x_k, y_k, closed curves through the points $x_k{}^0=\phi_k(t_0)$, $y_k{}^0=\psi_k(t_0)$. The hypothesis is that if we start from the development of F in powers of $x_k-x_k{}^0$, $y_k-y_k{}^0$, its analytical continuation along these curves back to $x_k{}^0$, $y_k{}^0$ is to yield the same series as at starting.

† The coefficients would not necessarily be periodic if F were not single-valued in the neighbourhood of the solution (5.2).

in the ξ_k, η_k, we obtain the variational equations of the solution (5.2). By means of a homogeneous linear contact transformation with coefficients periodic in t, say

$$\xi_k = \theta_k(\zeta_r, \omega_r, t), \qquad \eta_k = \chi_k(\zeta_r, \omega_r, t) \tag{5.5}$$

we may reduce these variational equations to a simple " normal " form*—

$$\frac{d\zeta_k}{dt} = \frac{\partial G_2}{\partial \omega_k}, \qquad \frac{d\omega_k}{dt} = -\frac{\partial G_2}{\partial \zeta_k}, \qquad (k = 1, 2); \tag{5.6}$$

if we write $\nu = 2\pi\iota/T$, and $0, 0, \pm \lambda$ are the exponents of the solution (5.2), the form of G_2 is in the various possible cases† as follows :—

$$\text{if } \lambda \neq \tfrac{1}{2}\nu,\ \lambda \neq 0, \qquad G_2 = \lambda\zeta_1\omega_1 - \tfrac{1}{2}a\zeta_2^2; \tag{5.7}$$

$$\text{if } \lambda = \tfrac{1}{2}\nu, \qquad G_2 = \tfrac{1}{2}\nu\zeta_1\omega_1 - \tfrac{1}{2}a\zeta_2^2 + \tfrac{1}{2}b\zeta_1^2 e^{-\nu t}; \tag{5.8}$$

$$\text{if } \lambda = 0, \qquad G_2 = a\zeta_2\omega_1 + \tfrac{1}{2}b\zeta_1^2 + c\zeta_1\zeta_2 + \tfrac{1}{2}d\zeta_2^2, \tag{5.9}$$

where a, b, c, d are constants. Now apply to (5.4) the transformation (5.5) and we obtain equations of the form

$$\frac{d\zeta_k}{dt} = \frac{\partial G}{\partial \omega_k}, \qquad \frac{d\omega_k}{dt} = -\frac{\partial G}{\partial \zeta_k}, \tag{5.10}$$

in which $G = G_2 + G'$, and G' is a power series in the ζ_k, ω_k with periodic coefficients, whose terms of lowest degree are at least cubic.

We assume that all the periodic functions of which there has been mention are expansible as Laurent series in $e^{2\pi\iota t/T}$, *i.e.*, in $e^{\nu t}$. Since F does not involve t we may add to t an arbitrary constant ε in the solution (5.2), and wherever t appears subsequently; writing $\gamma = e^{\nu\varepsilon}$ the periodic functions are thus Laurent series in $\gamma e^{\nu t}$, where γ is an arbitrary constant. With the convergence of these series we are not at present concerned.‡ With a slight change of notation we write, then, for the linear contact transformation which results from (5.3), (5.5)

$$x_k = \theta_k(\zeta_r, \omega_r, \gamma e^{\nu t}), \ y_k = \chi_k(\zeta_r, \omega_r, \gamma e^{\nu t}) \tag{5.11}$$

and (5.8) becomes

$$G_2 = \tfrac{1}{2}\nu\zeta_1\omega_1 - \tfrac{1}{2}a\zeta_2^2 + \tfrac{1}{2}b\zeta_1^2(\gamma e^{\nu t})^{-1}. \tag{5.12}$$

The generating solution (5.2) is obtained from (5.11) by putting $\zeta_k = \omega_k = 0$. We

* CHERRY, 'Proc. Lond. Math. Soc.,' vol. 26, p. 211 (1927), " On the transformation of Hamiltonian systems of linear differential equations with constant or periodic coefficients."

† Only these cases are possible because by definition we may always suppose that the real part of λ/ν lies between 0 and $\frac{1}{2}$.

‡ The justification of the hypotheses that have been made concerning the single-valuedness of F and the expansibility of the periodic functions, and that will be made concerning the convergence of the series, will be considered in §§ 10, 11, below.

call (5.11) the *normalising contact-transformation* for the neighbourhood of this solution, and (5.10) the corresponding *normal form* of the equations (5.1).

Formal Solution of the Equations.—It has been shown by the author* how to construct for (5.10) a general solution of the following form: the ζ_k, ω_k are power series in four arguments—

$$\alpha_1 e^{\lambda t}, \qquad \beta_1 e^{-\lambda t}, \qquad \alpha_2, \ \beta_2,$$

with coefficients which are Laurent series in $\gamma e^{\nu t}$ with coefficients polynomial in t; the series begin with terms of the first degree; here γ remains an arbitrary constant, and the α_k, β_k are arbitrary constants whose Lagrange brackets have the values

$$[\alpha_1, \beta_1]_{\zeta\omega} = [\alpha_2, \beta_2]_{\zeta\omega} = 1$$
$$[\alpha_1, \alpha_2]_{\zeta\omega} = [\alpha_1, \beta_2]_{\zeta\omega} = [\beta_1, \alpha_2]_{\zeta\omega} = [\beta_1, \beta_2]_{\zeta\omega} = 0.$$

Inserting this solution in (5.11) we have for the x_k, y_k a general solution of the same form, say—

$$\left.\begin{aligned} x_k &= f_k'\,(\alpha_1 e^{\lambda t}, \beta_1 e^{-\lambda t}, \alpha_2, \beta_2, \gamma e^{\nu t}, t) \\ y_k &= g_k'\,(\alpha_1 e^{\lambda t}, \beta_1 e^{-\lambda t}, \alpha_2, \beta_2, \gamma e^{\nu t}, t) \end{aligned}\right\}, \qquad \dots\dots \quad (5.13)$$

and since (5.11) is a contact transformation the relations just written lead to

$$\left.\begin{aligned} &[\alpha_1, \beta_1]_{f'g'} = [\alpha_2, \beta_2]_{f'g'} = 1 \\ &[\alpha_1, \alpha_2]_{f'g'} = [\alpha_1, \beta_2]_{f'g'} = [\beta_1, \alpha_2]_{f'g'} = [\beta_1, \beta_2]_{f'g'} = 0 \end{aligned}\right\} \quad \dots \quad (5.14)$$

Second Transformation.—Put $t = 0$ throughout (5.13) and then replace γ wherever it occurs by $\gamma' e^{\sigma t}$; this yields a transformation, say—

$$\left.\begin{aligned} x_k &= f_k\,(\alpha_1, \beta_1, \alpha_2, \beta_2, \gamma' e^{\sigma t}) \\ y_k &= g_k\,(\alpha_1, \beta_1, \alpha_2, \beta_2, \gamma' e^{\sigma t}) \end{aligned}\right\}, \qquad \dots\dots \quad (5.15)$$

to new variables α_1, β_1, α_2, β_2, depending on two arbitrary parameters γ', σ and on t, as well as on the α_k, β_k.

We first note that (5.15) is a contact transformation, for on forming the Lagrange brackets of pairs of the α_k, β_k we do not differentiate with respect to t or γ, so that, for instance—

$$\left[\frac{\partial}{\partial\alpha_1}\{f'_k(\alpha_1 e^{\lambda t}, \beta_1 e^{-\lambda t}, \alpha_2, \beta_2, \gamma e^{\nu t}, t)\}\right]_{t=0,\,\gamma=\gamma' e^{\sigma t}}$$
$$= \frac{\partial}{\partial\alpha_1}[f'_k\,(\alpha_1 e^{\lambda t}, \beta_1 e^{-\lambda t}, \alpha_2, \beta_2, \gamma e^{\nu t}, t)_{t=0,\,\gamma=\gamma' e^{\sigma t}}] = \frac{\partial}{\partial\alpha_1}\{f_k(\alpha_1, \beta_1, \alpha_2, \beta_2, \gamma' e^{\sigma t})\},$$

and

$$[\alpha_1, \beta_1]_{fg} = \{[\alpha_1, \beta_1]_{f'g'}\}_{t=0,\,\gamma=\gamma' e^{\sigma t}} = 1\,;$$

* 'Proc. Lond. Math. Soc.,' vol. 27, p. 151 (1927), "On the solution of Hamiltonian systems of differential equations in the neighbourhood of a singular point."

thus, from (5.14) we deduce

$$\left.\begin{array}{c}[\alpha_1, \beta_1]_{fg} = [\alpha_2, \beta_2]_{fg} = 1 \\ [\alpha_1, \alpha_2]_{fg} = [\alpha_1, \beta_2]_{fg} = [\beta_1, \alpha_2]_{fg} = [\beta_1, \beta_2]_{fg} = 0\end{array}\right\},$$

which are the conditions that (5.15) should be a contact transformation.

From § 3 the Hamiltonian function of the transformed system is F + M, where M is defined by the consistent equations

$$\frac{\partial M}{\partial \alpha_k} = [\alpha_k, t]_{fg}, \qquad \frac{\partial M}{\partial \beta_k} = [\beta_k, t]_{fg}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5.16)$$

First consider the expression for F. This is obtainable by substituting from (5.13) to express F in terms of γ, t and the α_k, β_k, and then putting $t = 0$, $\gamma = \gamma' e^{\sigma t}$. On making this substitution F becomes a series of the form

$$F = \Sigma\, (\alpha_1 e^{\lambda t})^{a_1} (\beta_1 e^{-\lambda t})^{b_1} \alpha_2{}^{a_2} \beta_2{}^{b_2} (\gamma e^{\nu t})^c \times (\text{polynomial in } t),$$

where the a_k, b_k are non-negative integers and c is an integer (positive, negative or zero) and since F is an integral of the equations (5.1), t must disappear from this series for all values of the arbitrary γ, α_k, β_k. Thus the polynomial in t must vanish identically unless

$$\lambda\,(a_1 - b_1) + \nu c = 0, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5.17)$$

and must reduce to a constant if this relation is satisfied; F then becomes a series $\Sigma A_{a_1 b_1 a_2 b_2 c}\, \alpha_1{}^{a_1} \beta_1{}^{b_1} \alpha_2{}^{a_2} \beta_2{}^{b_2} \gamma^c$, where A is a numerical* coefficient, and for each term the indices satisfy (5.17). Putting $t = 0$, $\gamma = \gamma' e^{\sigma t}$, we have

$$F \equiv \Sigma A_{a_1 b_1 a_2 b_2 c}\, \alpha_1{}^{a_1} \beta_1{}^{b_1} \alpha_2{}^{a_2} \beta_2{}^{b_2} (\gamma' e^{\sigma t})^c. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5.18)$$

We now show that M is a series of similar form. From the form of (5.15) we have

$$\frac{\partial f_k}{\partial t} = \sigma\gamma' \frac{\partial f_k}{\partial \gamma'}, \qquad \frac{\partial g_k}{\partial t} = \sigma\gamma' \frac{\partial g_k}{\partial \gamma'},$$

so that

$$[\alpha_1, t]_{fg} = \sigma\gamma' [\alpha_1, \gamma']_{fg}.$$

Now t occurs as a constant parameter throughout the performance of differentiations with respect to γ' or α_1; moreover, the operation $\gamma' (\partial/\partial\gamma')$ on any term leaves its degree in γ' unaltered. Bearing in mind the relation between the functions f_k, g_k and f'_k, g'_k, we have therefore

$$\gamma' [\alpha_1, \gamma']_{fg} = \{\gamma\, [\alpha_1, \gamma]_{f'g'}\}_{t=0,\, \gamma = \gamma' e^{\sigma t}}.$$

* *I.e.*, its value depends on the coefficients in $F(x_k, y_k)$, but not on the arbitrary integration-constants α_1, β_1, α_2, β_2, γ.

Now consider the expression in series for $\gamma\,[\alpha_1, \gamma]_{f'g'}$; it has the form

$$(1/\alpha_1) \times (\text{series in the arguments } \alpha_1 e^{\lambda t},\ \beta_1 e^{-\lambda t},\ \alpha_2,\ \beta_2,\ \gamma e^{\nu t},\ t),$$

and since α_1, γ are arbitrary integration-constants t must disappear identically from this expression.* Hence as above

$$\gamma\,[\alpha_1, \gamma]_{f'g'} = (1/\alpha_1)\,.\,\Sigma\,B_{a_1 b_1 a_2 b_2 c}\,\alpha_1{}^{a_1}\beta_1{}^{b_1}\alpha_2{}^{a_2}\beta_2{}^{b_2}\gamma^c,$$

where B is a numerical coefficient, and the summation is over values of the indices satisfying (5.17). This gives

$$\begin{aligned}[\alpha_1, t]_{fg} &= \sigma\,\{\gamma\,[\alpha_1, \gamma]_{f'g'}\}_{t=0,\ \gamma=\gamma' e^{\sigma t}} \\ &= (\sigma/\alpha_1)\,.\,\Sigma\,B_{a_1 b_1 a_2 b_2 c}\,\alpha_1{}^{a_1}\beta_1{}^{b_1}\alpha_2{}^{a_2}\beta_2{}^{b_2}\,(\gamma' e^{\sigma t})^c.\end{aligned}$$

There are similar expressions for $[\beta_1, t]_{fg}$, etc., and from (5.16) M must have the form

$$M \equiv \sigma\,\Sigma\,C_{a_1 b_1 a_2 b_2 c}\,\alpha_1{}^{a_1}\beta_1{}^{b_1}\alpha_2{}^{a_2}\beta_2{}^{b_2}\,(\gamma' e^{\sigma t})^c. \quad \ldots \quad (5.19)$$

It is convenient now to re-write the contact transformation (5.15), omitting the accent from γ' and calling the new variables z_k, u_k instead of α_k, β_k. We have the result that *there is a formal contact transformation*

$$x_k = f_k\,(z_r, u_r, \gamma e^{\sigma t}), \qquad y_k = g_k\,(z_r, u_r, \gamma e^{\sigma t}) \qquad (k = 1, 2), \quad \ldots \quad (5.20)$$

in which the f_k, g_k *are power series in the* z_r, u_r, *with coefficients which are Laurent series in* $\gamma e^{\sigma t}$, *whereby the equations* (5.1) *become*

$$\frac{dz_k}{dt} = \frac{\partial K}{\partial u_k}, \qquad \frac{du_k}{dt} = -\frac{\partial K}{\partial z_k} \qquad (k = 1, 2), \quad \ldots \quad (5.21)$$

K *being a series*

$$K \equiv F + M \equiv \Sigma\,(A_{a_1 b_1 a_2 b_2 c} + \sigma C_{a_1 b_1 a_2 b_2 c})\,z_1{}^{a_1}u_1{}^{b_1}z_2{}^{a_2}u_2{}^{b_2}\,(\gamma e^{\sigma t})^c, \quad \ldots \quad (5.22)$$

in which, for each term, the indices are integers satisfying

$$\left.\begin{array}{c}\lambda\,(a_1 - b_1) + \nu c = 0 \\ a_1 \geq 0, \quad b_1 \geq 0, \quad a_2 \geq 0, \quad b_2 \geq 0\end{array}\right\}; \quad \ldots \quad (5.23)$$

the transformation involves the arbitrary parameters γ, σ. We observe that these parameters appear in K in the manner explicitly shown in (5.22) only; the A's and C's have numerical values which depend only on the numerical values of the coefficients in the

* Whittaker, "Analytical Dynamics," § 146.

original Hamiltonian function $F(x_k, y_k)$, and upon the particular generating solution (5.2). The original periodic solution from which we started is obtained from (5.20) if we put $\sigma = \nu$, $z_1 = u_1 = z_2 = u_2 = 0$.

Form of the Series for K.—We enquire first as to the leading terms in the series. When $\sigma = \nu$ we obtain (5.20) from (5.13) by putting $t = 0$, except in the argument $\gamma e^{\nu t}$, and replacing the α_k, β_k by z_k, u_k, a change which does not affect (5.11); hence we may obtain the relations connecting the ζ_k, ω_k with the z_k, u_k by treating the general solution for the ζ_k, ω_k as we treated (5.13). A first approximation to this general solution is given by solving (5.6). If $\lambda \neq \frac{1}{2}\nu$, $\lambda \neq 0$, the solution is

$$\zeta_1 = \alpha_1 e^{\lambda t}, \qquad \omega_1 = \beta_1 e^{-\lambda t}, \qquad \zeta_2 = \alpha_2, \qquad \omega_2 = \beta_2 + a\alpha_2 t,$$

and hence, putting $t = 0$ and replacing the α_k, β_k by z_k, u_k, we have the first approximation

$$\zeta_1 = z_1, \qquad \omega_1 = u_1, \qquad \zeta_2 = z_2, \qquad \omega_2 = u_2;$$

the same first approximation is found also if $\lambda = \frac{1}{2}\nu$ or $\lambda = 0$. Hence, when $\sigma = \nu$ the leading terms in the series K are of the same form as those in the series G, viz., are quadratic and of the form given by (5.7), (5.9), (5.12) in the various cases.

Now suppose $\sigma \neq \nu$. Then $x_k = \theta_k(0, 0, \gamma e^{\sigma t})$, $y_k = \chi_k(0, 0, \gamma e^{\sigma t})$ is not a solution of (5.1),* so $z_1 = u_1 = z_2 = u_2 = 0$ is not a solution of (5.21), and K must contain terms linear in the z_k, u_k. The coefficients in K are linear functions of the arbitrary parameter σ; hence K contains the linear terms

$$(\sigma - \nu)(c_1 z_1 + d_1 u_1 + c_2 z_2 + d_2 u_2), \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5.24)$$

where at least one of the constants c_1, d_1, c_2, d_2 is non-zero.

We now proceed to consider the implications of the relation (5.23).

CASE I: λ, ν *incommensurable*.—Here (5.23) requires $a_1 = b_1$, $c = 0$; hence z_1, u_1 occur in K only through the argument $z_1 u_1$, $= v$ say, and K does not depend upon the argument $\gamma e^{\sigma t}$. In (5.24) we must then have $c_1 = d_1 = 0$. We may write K as a power series in the arguments v, z_2, u_2, $(\sigma - \nu)$—no powers of $(\sigma - \nu)$ higher than the first occurring, of course—

$$K \equiv (\sigma - \nu)(c_2 z_2 + d_2 u_2) + \lambda v - \tfrac{1}{2} a z_2^2 + c_3(\sigma - \nu) v + c_4 z_2 v + c_5 u_2 v + \ldots, \quad (5.25)$$

and here at least one of c_2, d_2 is non-zero.

In the remaining cases λ, ν are commensurable, say $\lambda/\nu = \lambda_0/\nu_0$, where λ_0, ν_0 are

* When substituted in (5.1) this evidently makes the left-hand sides greater than the right in the ratio $\sigma : \nu$ since it makes them equal when $\sigma = \nu$. The statement in the text holds then, provided the substitution in question does not make both sides vanish, which is not so since (5.2) is not an equilibrium solution of (5.1).

integers mutually prime. By definition (§ 4) we have (λ/ν being real) $0 \leq \lambda_0/\nu_0 \leq \frac{1}{2}$, and thus $0 \leq \lambda_0 \leq \frac{1}{2}\nu_0$, $\nu_0 \geq 2$. We shall later have to distinguish the following cases :—

$$\left.\begin{array}{lll} \text{Case II}: & \nu_0 > 4 & \\ \text{Case III}: & \nu_0 = 4, & \lambda_0 = 1 \\ \text{Case IV}: & \nu_0 = 3, & \lambda_0 = 1 \\ \text{Case V}: & \nu_0 = 2, & \lambda_0 = 1 \end{array}\right\}. \quad \text{Here (5.23) requires } a_1 - b_1 = m\nu_0, \quad c = -m\lambda_0,$$

where m is an integer, and the typical term in (5.22) can be written in either of the forms

$$(\mathrm{A} + \sigma\mathrm{C})\,(z_1u_1)^{a_1}\,(u_1^{\nu_0}\gamma^{\lambda_0}e^{\sigma\lambda_0 t})^{-m}z_2^{\,a_2}u_2^{\,b_2},$$

$$(\mathrm{A} + \sigma\mathrm{C})\,(z_1u_1)^{b_1}\,(z_1^{\nu_0}\gamma^{-\lambda_0}e^{-\sigma\lambda_0 t})^{m}z_2^{\,a_2}u_2^{\,b_2},$$

of which the first is appropriate when m is negative and the second when m is positive. Hence K can be written as a power series in the six arguments v, w, w', z_2, u_2, $(\sigma - \nu)$—no powers above the first of $(\sigma - \nu)$ occurring—where

$$v = z_1u_1, \qquad w = z_1^{\nu_0}(\gamma e^{\sigma t})^{-\lambda_0}, \qquad w' = u_1^{\nu_0}(\gamma e^{\sigma t})^{\lambda_0}; \quad . \; . \; . \; . \; . \quad (5.26)$$

these are equivalent to five arguments only since $ww' = v^{\nu_0}$, but it is generally advantageous to keep the six to avoid the occurrence of negative powers in K. In all the Cases II–V, v, w, w' are at least quadratic in z_1, u_1, so in (5.24) we must have $c_1 = d_1 = 0$. We write then

$$\begin{aligned} \mathrm{K} \equiv {} & (\sigma - \nu)\,(c_2z_2 + d_2u_2) + \lambda v - \tfrac{1}{2}az_2^{\,2} + c_3\,(\sigma - \nu)\,v + c_4z_2v + c_5u_2v \\ & + c_6w + c_7w' + (\sigma - \nu)\,(c_8w + c_9w') + \tfrac{1}{2}c_{10}v^2 + c_{11}vw + c_{12}vw' \\ & + c_{13}wz_2 + c_{14}w'z_2 + c_{15}wu_2 + c_{16}w'u_2 + \tfrac{1}{2}c_{17}w^2 + \tfrac{1}{2}c_{18}w'^2 \\ & + c_{19}ww' + \ldots \qquad (5.27) \end{aligned}$$

The terms independent of $(\sigma - \nu)$ and quadratic in the z_k, u_k have been made of the form (5.7), as has been seen must be the case. If $\nu_0 = 2$, $\lambda_0 = 1$ (Case V), w and w' as well as v are quadratic in the z_k, u_k, and (5.12) shows that

$$c_6 = \tfrac{1}{2}b, \qquad c_7 = 0. \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (5.28)$$

Case VI: $\lambda = 0$.—Here (5.23) requires that $c = 0$ and places no restriction on a_1, b_1. Hence K is a power series in the z_k, u_k with coefficients linear in $(\sigma - \nu)$, the quadratic terms having the form (5.9) when $\sigma - \nu = 0$:

$$\mathrm{K} \equiv (\sigma - \nu)\,(c_1z_1 + d_1u_1 + c_2z_2 + d_2u_2) + az_2u_1 + \tfrac{1}{2}bz_1^{\,2} + cz_1z_2 + \tfrac{1}{2}dz_2^{\,2} + \ldots, \quad (5.29)$$

where at least one of c_1, d_1, c_2, d_2 is not zero.

Method by which Periodic Solutions are established.—Any solution of (5.21), when substituted in (5.20), yields a corresponding solution of (5.1). If the former solution is periodic and of parameter σ, *i.e.*, if the z_k, u_k are Laurent series in $e^{\sigma t}$, the latter solution is manifestly of the same nature; in particular, to an equilibrium solution for the z_k, u_k corresponds for the x_k, y_k such a periodic solution. Similarly to a periodic solution of parameter σ/m (m integral) for the z_k, u_k corresponds for the x_k, y_k a periodic solution of the same parameter. The fact that z_1, u_1, t are only involved in K through the composite arguments v, w, w' will enable us to find periodic solutions of (5.21), and therefore of (5.1), and the fact that in (5.20) σ is an arbitrary parameter will enable us to find continuous families of such solutions of continuously varying parameter.

Suppose

$$z_k = p_k\,(e^{\sigma t/m}),\ u_k = q_k\,(e^{\sigma t/m}) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5.30)$$

is a periodic solution of parameter σ/m, where m is an integer,* and

$$x_k = \mathrm{P}_k\,(e^{\sigma t/m}), \qquad y_k = \mathrm{Q}_k\,(e^{\sigma t/m}) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5.31)$$

the corresponding solution of (5.1). We form the variational equations of these solutions by substituting in (5.21), (5.1), respectively—

$$z_k = p_k + z'_k, \qquad u_k = q_k + u'_k$$

and

$$x_k = \mathrm{P}_k + x'_k, \qquad y_k = \mathrm{Q}_k + y'_k.$$

If we make these substitutions in (5.20) we obtain to the first order the x'_k, y'_k as homogeneous linear functions of the z'_k, u'_k, with periodic coefficients (parameter σ/m), whence from any solution for the z'_k, u'_k we can derive the corresponding solution for the x'_k, y'_k. If μ is an exponent of (5.30) there is a solution for the z'_k, u'_k of the form

$$z'_k,\ u'_k = e^{\mu t} \times \text{(periodic function of } t \text{ of parameter } \sigma/m),$$

and the corresponding solution for the x'_k, y'_k has evidently the same form with the *same* value of μ. Hence the exponents of (5.31) are the same as the exponents of (5.30), and we may find not only periodic solutions of (5.1) but also their exponents by dealing with the transformed equations (5.21).

For the formal solutions of (5.1) and the formal expressions for their exponents thus derived to be significant, it is essential that they should be specified by convergent series. This convergence will be established later (§§ 10, 11).

Real Transformations when F is a Real Function of the x_k, y_k.—The question of the reality of the transformation (5.20) becomes now of interest. Suppose that in the generating solution (5.2) the period T is real, so that the parameter ν is pure-imaginary, and that the x_k, y_k are real when t, ε are real; under these conditions $|\gamma| = |e^{\nu\varepsilon}| = 1$, and (5.2) is a real solution.

If a real periodic solution possesses an unreal exponent it must also possess the conjugate exponent; since two exponents are always zero and the other two have their sum zero,

* A solution of parameter σ is included as a particular case.

the only possibilities concerning the non-zero exponents $\pm\lambda$ of the generating solution are (i) λ real, (ii) λ pure-imaginary, (iii) $\lambda = \frac{1}{2}\nu + R$, where R is real and non-zero.* The following results may be proved in the three cases by considering the mode in which the transformation (5.20) is obtained: the proof is omitted for lack of space.

For a periodic solution to have a real period its parameter σ must be pure-imaginary, and we are interested therefore in solutions for which the x_k, y_k are real when σ is pure-imaginary and t and its additive arbitrary constant $(\log\gamma)/\sigma$ are real (so that $|\gamma| = 1$).

When λ *is real* we may always suppose† that (5.20) is a real contact transformation, viz., it gives the x_k, y_k as real functions of the z_k, u_k *when* σ, ν *are pure-imaginary, t is real and* $|\gamma| = 1$. Since λ/ν is unreal the circumstances are those of Case I, and K is a series in v, z_2, u_2, $\iota(\sigma - \nu)$ with real coefficients.

When λ *is pure-imaginary and* $0 < |\lambda| < \frac{1}{2}|\nu|$, we may always suppose that (5.20), when combined with either

$$\text{(Case A)} \qquad \sqrt{2}z_1 = z'_1 + \iota u'_1, \qquad \sqrt{2}u_1 = u'_1 + \iota z'_1 \quad . \quad . \quad . \quad (5.32)$$

$$\text{or (Case B)} \qquad \sqrt{2}z_1 = u'_1 + \iota z'_1, \qquad \sqrt{2}u_1 = -z'_1 - \iota u'_1 \quad . \quad . \quad . \quad (5.33)$$

gives the x_k, y_k as real functions of z'_1, u'_1, z_2, u_2 when σ, ν are pure-imaginary, t is real and $|\gamma| = 1$; the subsidiary transformations (5.32), (5.33) are, of course, contact transformations. K is then a real function of z'_1, u'_1, z_2, u_2, $\iota(\sigma - \nu)$.

When $\lambda = \frac{1}{2}\nu$ we similarly combine (5.20) with

$$z_1 = Z_1 (\gamma e^{\sigma t})^{\frac{1}{2}}, \qquad u_1 = U_1 (\gamma e^{\sigma t})^{-\frac{1}{2}}, \quad . \quad . \quad . \quad (5.34)$$

and obtain the x_k, y_k as real functions of Z_1, U_1, z_2, u_2. Moreover,

$$K = \tfrac{1}{2}\sigma z_1 u_1 + L \quad . \quad . \quad . \quad (5.35)$$

where L is a real function of Z_1, U_1, z_2, u_2, $\iota(\sigma - \nu)$.

When $\lambda = 0$, we may always suppose that (5.20) is a real transformation, and K is a power series in z_1, u_1, z_2, u_2, $\iota(\sigma - \nu)$ with real coefficients.

When $\lambda = \frac{1}{2}\nu + R$ (R real), we may always suppose that (5.20), when combined with (5.34), gives the x_k, y_k as real functions of Z_1, U_1, z_2, u_2, $\iota(\sigma - \nu)$.

§ 6. *Periodic Solutions when* λ, ν *are Incommensurable.*

In the equations (5.21) we have $K \equiv K(v, z_2, u_2, \sigma - \nu)$, where $v = z_1 u_1$, and K is an integral of the equations since it does not involve t. Moreover, since they are derived

* By their original definition the exponents are indeterminate to an arbitrary integral multiple of ν, so an exponent with imaginary part $\frac{1}{2}\nu$ may be regarded as its own conjugate.

† The process of formation of (5.20) does not determine it uniquely, *i.e.*, there is a family of contact transformations, any one of which will bring the equations (5.1) to the form (5.21). Only certain transformations of this family will be real.

by transformation from (5.1), which possess F as an integral, F is also—when expressed in terms of the z_k, u_k—an integral of (5.21). Hence M, $=$ K $-$ F, is also an integral* of (5.21), and we have identically

$$\frac{\partial M}{\partial z_1}\frac{\partial K}{\partial u_1}-\frac{\partial M}{\partial u_1}\frac{\partial K}{\partial z_1}+\frac{\partial M}{\partial z_2}\frac{\partial K}{\partial u_2}-\frac{\partial M}{\partial u_2}\frac{\partial K}{\partial z_2}=0.$$

Now M is that part of K which has σ as a factor, viz., from (5.25)

$$\begin{aligned} M &= \sigma\,(c_2 z_2 + d_2 u_2 + c_3 v + \ldots) \\ &= \sigma\phi\,(z_2, u_2, v) \qquad \text{say;} \end{aligned}$$

and since K, M involve z_1, u_1 through the argument v, $= z_1 u_1$ only, the identity just written reduces to

$$\frac{\partial \phi}{\partial z_2}\frac{\partial K}{\partial u_2}-\frac{\partial \phi}{\partial u_2}\frac{\partial K}{\partial z_2}=0. \qquad (6.1)$$

Here K is the series (5.25) and

$$\phi = c_2 z_2 + d_2 u_2 + c_3 v + \ldots, \qquad (6.2)$$

at least one of c_2, d_2 being not zero. Thus (6.1) gives

$$(c_2+\ldots)\,(c_5 v+\ldots) - (d_2+\ldots)\,(-a z_2 + c_4 v + \ldots) \equiv 0,$$

whence, from the coefficient of z_2,

$$a d_2 = 0.$$

Thus if $a \neq 0$ we must have $d_2 = 0$, $c_2 \neq 0$, while if $a = 0$ we may always suppose that $c_2 \neq 0$; for if $c_2 = 0$ we have $d_2 \neq 0$, and a contact transformation

$$z_2 = u'_2, \qquad u_2 = -z'_2$$

brings (5.25) to a series of the same form in which the coefficient of z'_2 is not zero.

Suppose then that $c_2 \neq 0$. From (6.2) we can express z_2 as a power series in ϕ, u_2, v, and hence K may be expressed as a power series in ϕ, u_2, v, with coefficients linear in $(\sigma - \nu)$; the identity (6.1) then implies that u_2 disappears identically from this series, *i.e.*, we have

$$\begin{aligned} K &= (\sigma - \nu)\,\phi + \lambda v - \tfrac{1}{2} a \phi^2 / c_2^2 + \ldots \\ &\equiv K'\,(\phi, v, \sigma - \nu) \qquad \text{say.} \end{aligned}$$

* That M is an integral may be shown directly as a consequence of the fact that the transformation (5.20) gives for F and M functions of the z_k, u_k *not involving t.* For

$$0 = \frac{\partial F(z_k, u_k, t)}{\partial t} = \Sigma\left(\frac{\partial F}{\partial x_k}\frac{\partial x_k}{\partial t}+\frac{\partial F}{\partial y_k}\frac{\partial y_k}{\partial t}\right) = \Sigma\left(\frac{dx_k}{dt}\frac{\partial y_k}{\partial t}-\frac{dy_k}{dt}\frac{\partial x_k}{\partial t}\right)$$

$$= [t, t] + \Sigma\left\{[z_k, t]\frac{dz_k}{dt}+[u_k, t]\frac{du_k}{dt}\right\} = \Sigma\left(\frac{\partial M}{\partial z_k}\frac{dz_k}{dt}+\frac{\partial M}{\partial u_k}\frac{du_k}{dt}\right) = \frac{dM}{dt}.$$

We now investigate the equilibrium solutions of (5.21), to which, as we have seen, correspond periodic solutions of (5.1). These equilibrium solutions are the sets of values of the z_k, u_k which satisfy

$$\frac{\partial K}{\partial z_1} = \frac{\partial K}{\partial u_1} = \frac{\partial K}{\partial z_2} = \frac{\partial K}{\partial u_2} = 0,$$

viz.,

$$u_1\left(\frac{\partial K'}{\partial v} + \frac{\partial K'}{\partial \phi}\frac{\partial \phi}{\partial v}\right) = z_1\left(\frac{\partial K'}{\partial v} + \frac{\partial K'}{\partial \phi}\frac{\partial \phi}{\partial v}\right) = \frac{\partial K'}{\partial \phi}\frac{\partial \phi}{\partial z_2} = \frac{\partial K'}{\partial \phi}\frac{\partial \phi}{\partial u_2} = 0.$$

Here $\frac{\partial K'}{\partial v} = \lambda + \dots$ and $\frac{\partial \phi}{\partial z_2} = c_2 + \dots$, and neither vanishes in the neighbourhood of the origin $\sigma - \nu = z_1 = u_1 = z_2 = u_2 = 0$. Hence equilibrium solutions in this neighbourhood are given by

$$z_1 = u_1 = 0, \qquad \frac{\partial K'}{\partial \phi} = 0,$$

i.e.,

$$z_1 = u_1 = v = 0, \quad (\sigma - \nu) - a\phi/c_2^2 + \dots = 0. \qquad (6.3)$$

For any value of ϕ this determines a unique value of $(\sigma - \nu)$, vanishing with ϕ, such that $z_1 = u_1 = 0$ together with any values of z_2, u_2 satisfying (6.2) constitute an equilibrium solution for (5.21).

We now show that all such equilibrium solutions which belong to the same value of ϕ lead through (5.20) to *equivalent* periodic solutions for the x_k, y_k, *i.e.*, to solutions of which any one may be derived from any other by adding a constant to t. Let z_2, u_2 undergo increments dz_2, du_2 such that ϕ remains constant; the corresponding increment in x_1 is, since σ remains constant,

$$dx_1 = \frac{\partial x_1}{\partial z_2}\,dz_2 + \frac{\partial x_1}{\partial u_2}\,du_2,$$

where

$$\frac{\partial \phi}{\partial z_2}\,dz_2 + \frac{\partial \phi}{\partial u_2}\,du_2 = 0;$$

i.e.,

$$[z_2, t]\,dz_2 + [u_2, t]\,du_2 = 0,$$

since

$$\frac{\partial M}{\partial z_k} = [z_k, t], \qquad \frac{\partial M}{\partial u_k} = [u_k, t], \qquad M = \sigma\phi. \qquad (6.4)$$

Thus since $[z_2, t] = \sigma\frac{\partial \phi}{\partial z_2} = \sigma c_2 + \dots$, which does not vanish in the neighbourhood of the origin,

$$dx_1 = \frac{du_2}{[z_2, t]}\left\{\frac{\partial x_1}{\partial u_2}[z_2, t] - \frac{\partial x_1}{\partial z_2}[u_2, t]\right\}. \qquad (6.5)$$

Moreover, for the equilibrium solutions under consideration $z_1 = u_1 = 0$, so

$$[z_1, t] = \sigma\frac{\partial \phi}{\partial z_1} = \sigma u_1\frac{\partial \phi}{\partial v} = 0, \qquad [u_1, t] = \sigma z_1\frac{\partial \phi}{\partial v} = 0,$$

and (6.5) may be written

$$dx_1 = \frac{du_2}{[z_2, t]}\left\{\frac{\partial x_1}{\partial u_1}[z_1, t] - \frac{\partial x_1}{\partial z_1}[u_1, t] + \frac{\partial x_1}{\partial u_2}[z_2, t] - \frac{\partial x_1}{\partial z_2}[u_2, t]\right\},$$
$$= \frac{du_2}{[z_2, t]}\left\{\frac{\partial x_1}{\partial t}(x_1, y_1) - \frac{\partial y_1}{\partial t}(x_1, x_1) + \frac{\partial x_2}{\partial t}(x_1, y_2) - \frac{\partial y_2}{\partial t}(x_1, x_2)\right\}$$

on re-arrangement, where (x_1, y_1), etc., are Poisson brackets.* Hence, using the conditions that (5.20) is a contact transformation in the form involving Poisson brackets,† we have

$$dx_1 = \frac{\partial x_1}{\partial t}\frac{du_2}{[z_2, t]},$$

and similarly

$$dx_k = \frac{\partial x_k}{\partial t}\frac{du_2}{[z_2, t]}, \qquad dy_k = \frac{\partial y_k}{\partial t}\frac{du_2}{[z_2, t]} \quad (k = 1, 2).$$

The supposed alterations in z_2, u_2 have thus the same effect on the x_k, y_k as an alteration in t of the amount $dt = du_2/[z_2, t]$. By integration, finite alterations in z_2, u_2 such that ϕ remains constant are equivalent to the addition of a constant to t; if u_2 alters from u'_2 to u''_2 the equivalent change in t is $\int_{u'_2}^{u''_2} du_2/[z_2, t]$, the integrand being expressed as a function of u_2, ϕ by means of (6.2). *Thus, for the purpose of finding periodic solutions of* (5.1) *we lose no generality by seeking only equilibrium solutions of* (5.21) *for which* $u_2 = 0$, *with, of course,* $z_1 = u_1 = 0$, *and the equilibrium value of* z_2 *will be determined in terms of* $(\sigma - \nu)$ *by the relation* $\left(\frac{\partial K}{\partial z_2}\right)_{v=u_2=0} = 0$, *i.e.*,

$$c_2(\sigma - \nu) - az_2 + \ldots = 0. \qquad (6.6)$$

Explicitly the periodic solutions thus found are

$$x_k = f_k(0, 0, z_2, 0, \gamma e^{\sigma t}), \qquad y_k = g_k(0, 0, z_2, 0, \gamma e^{\sigma t}), \qquad (6.7)$$

where σ, z_2 are constants of which one is arbitrary, and γ is an arbitrary constant whose logarithm occurs additively with t. There is thus a continuous family of such periodic solutions in the neighbourhood of the generating solution (5.2), the family-parameter‡ being z_2; the parameter σ varies continuously over this family, taking the value ν for the generating solution, which corresponds to $z_2 = 0$. If $a \neq 0$, which is in general the case, we may solve (6.6) for z_2 as a power series in $(\sigma - \nu)$, and in (6.7), which are Laurent series in $\gamma e^{\sigma t}$, the coefficients are power series in $(\sigma - \nu)$; the family-parameter is then σ. If, however, $a = 0$, z_2 is obtained as a power series in a fractional power of $(\sigma - \nu)$—in

* $(x_1, y_1) = \frac{\partial(x_1, y_1)}{\partial(z_1, u_1)} + \frac{\partial(x_1, y_1)}{\partial(z_2, u_2)}$, etc.

† WHITTAKER, "Analytical Dynamics," § 131.

‡ *I.e.*, any quantity, a knowledge of whose value suffices to specify a particular member of the family.

general of $(\sigma - \nu)^{\frac{1}{2}}$—and the coefficients in question are power series of this nature* ; this case occurs when, as we move along the family, σ is stationary in value at the generating solution. It is now inappropriate to regard σ as the family-parameter, since by so doing we should give the family a false appearance of multiplicity.

It may be observed that there is a certain indeterminateness in the analytical expression for the family, owing to the arbitrariness of γ. Taking, for instance, the general case in which $a \neq 0$, we may replace γ by γP where P is any power series in $(\sigma - \nu)$, and will thus have the family represented by Laurent series in $\gamma e^{\sigma t}$, in which the coefficients are not the same power series in $(\sigma - \nu)$ as before. We shall later (§ 10) prove that of the different possible sets of series which represent the family, one definite set is convergent when $|\sigma - \nu|$ is sufficiently small, and, of course, any representation of the family which may be derived from this particular one by a *convergent* transformation of the sort just stated will also be convergent.

The Exponents of the Solution (6.7).—We form the variational equations of the equilibrium solution

$$z_1 = u_1 = 0, \qquad z_2 = z_2{}^0, \qquad u_2 = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6.8)$$

by substituting in (5.21) $z_2 = z_2{}^0 + \zeta_2$ and expanding to the first order in z_1, u_1, ζ_2, u_2. This gives

$$\frac{dz_1}{dt} = z_1 \left(\frac{\partial \mathrm{K}}{\partial v}\right)_0, \qquad \frac{du_1}{dt} = - u_1 \left(\frac{\partial \mathrm{K}}{\partial v}\right)_0,$$

$$\frac{d\zeta_2}{dt} = \zeta_2 \left(\frac{\partial^2 \mathrm{K}}{\partial z_2\, \partial u_2}\right)_0 + u_2 \left(\frac{\partial^2 \mathrm{K}}{\partial u_2{}^2}\right)_0, \qquad \frac{du_2}{dt} = - \zeta_2 \left(\frac{\partial^2 \mathrm{K}}{\partial z_2{}^2}\right)_0 - u_2 \left(\frac{\partial^2 \mathrm{K}}{\partial z_2\, \partial u_2}\right)_0, \quad (6.9)$$

where in the derivatives of K we are to substitute the equilibrium values (6.8). In virtue of the equilibrium condition $(\partial \mathrm{K}'/\partial \phi)_0 = 0$ we have

$$\left(\frac{\partial^2 \mathrm{K}}{\partial z_2\, \partial u_2}\right)_0 = \left(\frac{\partial \mathrm{K}'}{\partial \phi}\right)_0 \left(\frac{\partial^2 \phi}{\partial z_2\, \partial u_2}\right)_0 + \left(\frac{\partial^2 \mathrm{K}'}{\partial \phi^2}\right)_0 \left(\frac{\partial \phi}{\partial z_2}\right)_0 \left(\frac{\partial \phi}{\partial u_2}\right)_0 = \left(\frac{\partial^2 \mathrm{K}'}{\partial \phi^2}\right)_0 \left(\frac{\partial \phi}{\partial z_2}\right)_0 \left(\frac{\partial \phi}{\partial u_2}\right)_0,$$

and so on, and the last pair of equations reduce to

$$\frac{d\zeta_2}{dt} = \left(\frac{\partial^2 \mathrm{K}'}{\partial \phi^2}\right)_0 \left(\frac{\partial \phi}{\partial u_2}\right)_0 \left\{\zeta_2 \left(\frac{\partial \phi}{\partial z_2}\right)_0 + u_2 \left(\frac{\partial \phi}{\partial u_2}\right)_0\right\},$$

$$\frac{du_2}{dt} = - \left(\frac{\partial^2 \mathrm{K}'}{\partial \phi^2}\right)_0 \left(\frac{\partial \phi}{\partial z_2}\right)_0 \left\{\zeta_2 \left(\frac{\partial \phi}{\partial z_2}\right)_0 + u_2 \left(\frac{\partial \phi}{\partial u_2}\right)_0\right\},$$

having the solution

$$\zeta_2 = \alpha_2 \left(\frac{\partial^2 \mathrm{K}'}{\partial \phi^2}\right)_0 \left(\frac{\partial \phi}{\partial u_2}\right)_0^2 t + \beta_2 \left(\frac{\partial \phi}{\partial u_2}\right)_0,$$

$$u_2 = - \alpha_2 \left(\frac{\partial^2 \mathrm{K}'}{\partial \phi^2}\right)_0 \left(\frac{\partial \phi}{\partial z_2}\right)_0 \left(\frac{\partial \phi}{\partial u_2}\right)_0 t + \alpha_2 - \beta_2 \left(\frac{\partial \phi}{\partial z_2}\right)_0,$$

* As a very special case (6.6) may have every term factored by $(\sigma - \nu)$, so that in (6.7) we must put $\sigma = \nu$, z_2 remaining arbitrary.

where α_2, β_2 are arbitrary constants. Writing

$$m = \left(\frac{\partial K}{\partial v}\right)_0,$$

the first pair of equations have the solution

$$z_1 = \alpha_1 e^{mt}, \quad u_1 = \beta_1 e^{-mt},$$

where α_1, β_1 are arbitrary constants. The form of this solution shows that the exponents of (6.8) are 0, 0, $\pm m$, and in accordance with § 5, p. 156, above, the exponents of the solution (6.7) have these same values. From (5.25) we have

$$m = \lambda + c_3(\sigma - \nu) + c_4 z_2 + \dots,$$

and m varies continuously along the family, taking the value λ at the generating solution specified by $\sigma - \nu = z_2 = 0$; m is expressible as a power series in $(\sigma - \nu)$ or a fractional power of $(\sigma - \nu)$, according as $a \neq 0$ or $a = 0$; this series will later (§ 11) be proved convergent when $|\sigma - \nu|$ is sufficiently small.

Reality of the Periodic Solutions (6.7).—Suppose the generating solution is real and of real period. The conditions $z_1 = u_1 = 0$ with (5.32), (5.33) or (5.34) lead to $z'_1 = u'_1 = 0$ or $Z_1 = U_1 = 0$, respectively. In the condition $(\partial K/\partial z_2)_{v = u_2 = 0} = 0$ the left-hand side is then a power series in z_2, $\iota(\sigma - \nu)$ with real coefficients, and gives $\iota(\sigma - \nu)$ as a real function of z_2. Thus if z_2, t are real and $|\gamma| = 1$, the conditions stated at the end of § 5 for the reality of the x_k, y_k are satisfied whether λ be real, pure-imaginary or complex. *Hence, those solutions of the family for which the arbitrary z_2 is real and for which the arbitrary γ has modulus* 1 *have a real period, and are real when t is real.*

When λ is real, $(\partial K/\partial v)_0$ is a series in z_2, $\iota(\sigma - \nu)$ with real coefficients, becoming a real function of z_2 when we substitute for $(\sigma - \nu)$ from (6.6); hence the non-zero exponents of real solutions of the family are real in the neighbourhood of the generating solution.

When λ is pure-imaginary we have in Case A from (5.32)

$$v = z_1 u_1 = \tfrac{1}{2}\iota(z'^2_1 + u'^2_1),$$

so $(\partial K/\partial v)_0$ is pure-imaginary when z_2, $\iota(\sigma - \nu)$ are real; hence, for a real solution of the family the non-zero exponents are pure-imaginary in the neighbourhood of the generating solution. Case B is similar.

When $\lambda = \frac{1}{2}\nu + R$ (R real and non-zero) we have from (5.34), (5.35) that v is real and

$$\left(\frac{\partial K}{\partial v}\right)_0 = \tfrac{1}{2}\sigma + \left(\frac{\partial L}{\partial v}\right)_0,$$

and the second term is a series in z_2, $\iota(\sigma - \nu)$ with real coefficients; hence for a real solution of the family the non-zero exponents are complex.

Thus in all cases the exponents of real solutions of the family are of the same nature as those of the generating solution.

§ 7. *Periodic Solutions when* λ, ν *are Commensurable, and* λ ≠ 0.

In the equations (5.21) K is now a power series in the arguments v, w, w', z_2, u_2 defined by (5.26), with coefficients linear in $(\sigma - \nu)$; it involves t explicitly through the arguments w, w'. It is convenient to transform the equations again so as to obtain a system whose Hamiltonian function does not involve t. The required transformation is

$$z_1 = Z_1 (\gamma e^{\sigma t})^{\lambda_0/\nu_0}, \quad u_1 = U_1 (\gamma e^{\sigma t})^{-\lambda_0/\nu_0}, \quad z_2 = Z_2, \quad u_2 = U_2, \quad . \quad . \quad . \quad . \qquad (7.1)$$

which is evidently a contact transformation; the new Hamiltonian function is K + M′, where

$$\frac{\partial M'}{\partial Z_1} = [Z_1, t] = -\frac{\sigma\lambda_0 U_1}{\nu_0}, \qquad \frac{\partial M'}{\partial U_1} = [U_1, t] = -\frac{\sigma\lambda_0 Z_1}{\nu_0}, \qquad M' = -\frac{\sigma\lambda_0 Z_1 U_1}{\nu_0}.$$

We write

$$L = K + M' = K - \frac{\sigma\lambda_0 Z_1 U_1}{\nu_0} = K - \frac{\sigma\lambda_0 v}{\nu_0},$$

and have the equations

$$\frac{dZ_k}{dt} = \frac{\partial L}{\partial U_k}, \qquad \frac{dU_k}{dt} = -\frac{\partial L}{\partial Z_k} \qquad (k = 1, 2), \quad . \quad . \quad . \quad . \quad . \quad . \qquad (7.2)$$

in which L is a power series in v, w, w', z_2, u_2, $(\sigma - \nu)$ with

$$v = Z_1 U_1, \qquad w = z_1^{\nu_0} (\gamma e^{\sigma t})^{-\lambda_0} = Z_1^{\nu_0}, \qquad w' = U_1^{\nu_0}. \quad . \quad . \quad . \quad . \qquad (7.3)$$

For convenience of reference we write here the transformation from the x_k, y_k to the Z_k, U_k:

$$\begin{matrix} x_k \\ y_k \end{matrix} = \begin{matrix} f_k \\ g_k \end{matrix} (Z_1 \gamma^{\lambda_0/\nu_0} e^{\sigma\lambda_0 t/\nu_0}, U_1 \gamma^{-\lambda_0/\nu_0} e^{-\sigma\lambda_0 t/\nu_0}, Z_2, U_2, \gamma e^{\sigma t}), \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (7.4)$$

and we have L = F + N where N is defined by the consistent equations

$$\frac{\partial N}{\partial Z_k} = [Z_k, t], \qquad \frac{\partial N}{\partial U_k} = [U_k, t]. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (7.5)$$

Inserting the series (5.27) in the relation $L = K - \sigma\lambda_0 v/\nu_0$ we have

$$\begin{aligned} L \equiv\ & (\sigma - \nu)(c_2 Z_2 + d_2 U_2 + c'_3 v + c_8 w + c_9 w' + \ldots) \\ & - \tfrac{1}{2} a Z_2^2 + c_4 Z_2 v + c_5 U_2 v + c_6 w + c_7 w' + \tfrac{1}{2} c_{10} v^2 + c_{11} v w \\ & + c_{12} v w' + c_{13} w Z_2 + c_{14} w' Z_2 + c_{15} w U_2 + c_{16} w' U_2 \\ & + \tfrac{1}{2} c_{17} w^2 + \tfrac{1}{2} c_{18} w'^2 + c_{19} w w' + \ldots, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \end{aligned} \qquad (7.6)$$

where $c'_3 = c_3 - \lambda_0/\nu_0$.* This is not a general power series in v, w, w', Z_2, U_2, since

* The term λv of K combines with the added term $-\lambda_0 \sigma v/\nu_0$ to give $-\lambda_0 v (\sigma - \nu)/\nu_0$.

certain of the early terms have zero coefficients; also, as we shall see, the coefficients c are connected by certain relations.

The equations (7.2) possess the integral L, and also the integral F, since they are a transformation of (5.1); hence N, = L − F, is also an integral, and N is the part of L factored by σ:

$$N \equiv \sigma \left(c_2 Z_2 + d_2 U_2 + c'_3 v + c_8 w + c_9 w' + \ldots \right). \qquad (7.7)$$

If now in the identity

$$(N, L) \equiv \frac{\partial N}{\partial Z_1}\frac{\partial L}{\partial U_1} - \frac{\partial N}{\partial U_1}\frac{\partial L}{\partial Z_1} + \frac{\partial N}{\partial Z_2}\frac{\partial L}{\partial U_2} - \frac{\partial N}{\partial U_2}\frac{\partial L}{\partial Z_2} \equiv 0 \qquad (7.8)$$

we substitute the series (7.6), (7.7) we obtain a power series which must vanish identically; its coefficients are quadratic polynomials in the coefficients of the series (7.6), and their vanishing gives relations between these coefficients. The only such relation we shall require explicitly is $d_2 a = 0$, which results from the vanishing of the coefficient of Z_2. It shows that if $a \neq 0$ we must have $d_2 = 0$, and therefore $c_2 \neq 0$, while if $a = 0$ we may show as in § 6 that c_2 may always be supposed non-zero. We suppose, then, with no loss of generality, that $c_2 \neq 0$.

(i) *Existence of Periodic Solutions.*—We shall be concerned with equilibrium solutions of (7.2). To such a solution corresponds by (7.4) a periodic solution for the x_k, y_k, in general of parameter σ/ν_0; if, however, the equilibrium solution is of the form

$$Z_1 = U_1 = 0, \qquad Z_2 = Z_2^0, \qquad U_2 = U_2^0,$$

the corresponding periodic solution has the parameter σ. The equilibrium solutions of (7.2) are those sets of values of the Z_k, U_k which satisfy the conditions

$$\frac{\partial L}{\partial Z_1} = \frac{\partial L}{\partial U_1} = \frac{\partial L}{\partial Z_2} = \frac{\partial L}{\partial U_2} = 0. \qquad (7.9)$$

Now $\partial N/\partial Z_2 = \sigma c_2 + \ldots$ which does not vanish in the neighbourhood of the origin

$$Z_1 = U_1 = Z_2 = U_2 = \sigma - \nu = 0.$$

Hence the identity (7.8) shows that $\partial L/\partial U_2 = 0$ when the other three of conditions (7.9) are satisfied, and we may replace (7.9) by the *three* conditions

$$\frac{\partial L}{\partial Z_1} = \frac{\partial L}{\partial U_1} = \frac{\partial L}{\partial Z_2} = 0. \qquad (7.10)$$

Suppose these conditions solved for Z_1, U_1, Z_2 as functions of $(\sigma - \nu)$, U_2; on substitution in (7.4) we obtain a solution for the x_k, y_k depending on the three arbitrary constants σ, γ, U_2. Of these the logarithm of γ occurs additively with t; we shall show also that an alteration in U_2 is equivalent to the addition of a constant to t.

If U_2 undergoes an increment dU_2 while σ remains unaltered, the corresponding increments of Z_1, U_1, Z_2 are obtained by writing the conditions that the left-hand sides of the conditions (7.9) remain zero, viz. :—

$$\left.\begin{aligned}\frac{\partial^2 L}{\partial Z_k \partial Z_1} dZ_1 + \frac{\partial^2 L}{\partial Z_k \partial U_1} dU_1 + \frac{\partial^2 L}{\partial Z_k \partial Z_2} dZ_2 + \frac{\partial^2 L}{\partial Z_k \partial U_2} dU_2 = 0 \\ \frac{\partial^2 L}{\partial U_k \partial Z_1} dZ_1 + \frac{\partial^2 L}{\partial U_k \partial U_1} dU_1 + \frac{\partial^2 L}{\partial U_k \partial Z_2} dZ_2 + \frac{\partial^2 L}{\partial U_k \partial U_2} dU_2 = 0\end{aligned}\right\} (k = 1, 2). \quad . \quad (7.11)$$

But from the identity (7.8) we prove by differentiation with respect to Z_k or U_k that when (7.9) are satisfied

$$\left(\frac{\partial L}{\partial Z_k}, N\right) = \left(\frac{\partial L}{\partial U_k}, N\right) = 0, \quad (k = 1, 2) ;$$

hence the solution of (7.11) is

$$dZ_1 \Big/ \frac{\partial N}{\partial U_1} = - dU_1 \Big/ \frac{\partial N}{\partial Z_1} = dZ_2 \Big/ \frac{\partial N}{\partial U_2} = - dU_2 \Big/ \frac{\partial N}{\partial Z_2}.$$

The increment in x_1 corresponding to these increments in the Z_k, U_k is

$$\begin{aligned} dx_1 &= \Sigma_k \left(\frac{\partial x_1}{\partial Z_k} dZ_k + \frac{\partial x_1}{\partial U_k} dU_k\right) \\ &= - \Sigma_k \left(\frac{\partial x_1}{\partial Z_k}\frac{\partial N}{\partial U_k} - \frac{\partial x_1}{\partial U_k}\frac{\partial N}{\partial Z_k}\right) dU_2 \Big/ \frac{\partial N}{\partial Z_2}, \end{aligned}$$

since $\frac{\partial N}{\partial Z_2}$ does not vanish in the neighbourhood of the origin ; *i.e.*, from (7.5)

$$\begin{aligned} dx_1 &= - \frac{dU_2}{[Z_2, t]} \Sigma_k \left\{\frac{\partial x_1}{\partial Z_k} [U_k, t] - \frac{\partial x_1}{\partial U_k} [Z_k, t]\right\} \\ &= \frac{dU_2}{[Z_2, t]} \frac{\partial x_1}{\partial t}, \end{aligned}$$

on re-arrangement as in § 6, p. 160 above. Similarly,

$$dx_k = \frac{dU_2}{[Z_2, t]} \frac{\partial x_k}{\partial t}, \qquad dy_k = \frac{dU_2}{[Z_2, t]} \frac{\partial y_k}{\partial t} \quad (k = 1, 2),$$

which show that an increment dU_2 in U_2, σ remaining unaltered, is equivalent to the addition to t of the constant $dU_2/[Z_2, t]$. By integration a finite increment in U_2 is equivalent to the addition to t of the constant $\int dU_2/[Z_2, t]$, the integrand being expressed as a function of $(\sigma - \nu)$, U_2 by means of (7.10). We regard as equivalent two solutions, of which one differs from the other only by the addition of a constant to t ; for the purpose of finding periodic solutions for the x_k, y_k we therefore lose no generality by seeking only those equilibrium solutions for the Z_k, U_k in which $U_2 = 0$.

To take account of the restricted form in which L involves the arguments Z_1, U_1 we regard L as a power series in v, w, w', Z_2, U_2, $(\sigma - \nu)$. The first three arguments are by (7.3) connected by the relation

$$ww' - v^{\nu_0} = 0, \qquad (7.12)$$

but if we eliminate one of them from L we introduce negative or fractional powers of the others. We therefore write the conditions (7.10) in the form

$$U_1 \frac{\partial L}{\partial v} + \nu_0 Z_1^{\nu_0 - 1} \frac{\partial L}{\partial w} = Z_1 \frac{\partial L}{\partial v} + \nu_0 U_1^{\nu_0 - 1} \frac{\partial L}{\partial w'} = 0, \qquad (7.13)$$

$$\frac{\partial L}{\partial Z_2} = 0, \qquad (7.14)$$

where it is understood that in the derivatives of L we are to put $U_2 = 0$.

The first two of these are satisfied by

$$Z_1 = U_1 = 0$$

since $\nu_0 \geq 2$. If neither Z_1 nor U_1 is zero they are equivalent to

$$w \frac{\partial L}{\partial w} - w' \frac{\partial L}{\partial w'} = 0, \qquad w' \frac{\partial L}{\partial v} + \nu_0 v^{\nu_0 - 1} \frac{\partial L}{\partial w} = 0. \qquad (7.15)$$

If $U_1 = 0$, $Z_1 \neq 0$ we must have $\frac{\partial L}{\partial w} = \frac{\partial L}{\partial v} = 0$, and these three conditions, together with $\frac{\partial L}{\partial Z_2} = 0$, will in general have no solution for Z_1, U_1, Z_2 when σ remains arbitrary. There are thus in general only two sorts of equilibrium solutions:

(i) A solution

$$Z_1 = U_1 = U_2 = 0, \; Z_2 = Z_2^0 \qquad (7.16)$$

in which Z_2^0, σ are connected by the relation

$$\left(\frac{\partial L}{\partial Z_2}\right)_{Z_1 = U_1 = U_2 = 0, \quad Z_2 = Z_2^0} = 0. \qquad (7.17)$$

(ii) A solution in which $U_2 = 0$ and v, w, w', Z_2, σ are connected by the relations (7.12) (7.14) and (7.15), Z_1 and U_1 being then given by

$$Z_1^{\nu_0} = w, \qquad U_1^{\nu_0} = w', \qquad Z_1 U_1 = v. \qquad (7.18)$$

First consider the solution (7.16). Inserting the series (7.6) for L, the condition (7.17) has the form

$$c_2(\sigma - \nu) - aZ_2 + \ldots = 0,$$

and since $c_2 \neq 0$ we may solve for $(\sigma - \nu)$ as power series in Z_2. Inserting this in (7.4) we have for the x_k, y_k a continuous family of periodic solutions

$$\begin{matrix} x_k \\ y_k \end{matrix} = \begin{matrix} f_k \\ g_k \end{matrix} (0, 0, Z_2, 0, \gamma e^{\sigma t}), \qquad (7.19)$$

the family-parameter being the arbitrary Z_2. The parameter σ of a solution varies continuously as we pass along the family. The generating solution (5.2) belongs to the family, being given by $Z_2 = \sigma - \nu = 0$. If $a \neq 0$ we may solve (7.17) for Z_2 as a power series in $(\sigma - \nu)$, but if $a = 0$, Z_2 becomes a power series in a fractional power of $(\sigma - \nu)$; in the latter case σ is stationary in value at the generating solution. Thus *in general* the coefficients in the Laurent series (7.19) are expressible as power series in $(\sigma - \nu)$. The family of periodic solutions thus found is analogous to the family found in § 6, and will be called Family I.

Now consider solutions in which Z_1, U_1 do not vanish. It is easily seen that to any definite set of values of v, w, w', Z_2, $(\sigma - \nu)$ satisfying (7.12), (7.14) and (7.15) corresponds essentially only *one* periodic solution for the x_k, y_k in spite of the multiple-valuedness of Z_1, U_1 as given by (7.18). For since we must have $Z_1 U_1 = v$ the most general solution of (7.18) is

$$Z_1 = w^{1/\nu_0} e^{2\pi\iota n/\nu_0}, \qquad U_1 = v w^{-1/\nu_0} e^{-2\pi\iota n/\nu_0},$$

where a definite determination is taken for w^{1/ν_0} and n is any integer, and the corresponding solution for the x_k, y_k is

$$\begin{matrix} x_k \\ y_k \end{matrix} = \begin{matrix} f_k \\ g_k \end{matrix} \{w^{1/\nu_0} e^{2\pi\iota n/\nu_0} (\gamma e^{\sigma t})^{\lambda_0/\nu_0}, \qquad v w^{-1/\nu_0} e^{-2\pi\iota n/\nu_0} (\gamma e^{\sigma t})^{-\lambda_0/\nu_0}, \qquad Z_2, 0, \gamma e^{\sigma t}\}. \quad (7.20)$$

Since λ_0, ν_0 are relatively prime we can, given any two integers n', n'', find integers p, q such that $n' - n'' = p\lambda_0 + q\nu_0$, and the solution (7.20) with $n = n'$ is changed into that with $n = n''$ if we replace t by $t - 2\pi p\iota/\sigma$.

We proceed then to consider the solution of the conditions (7.12), (7.14), (7.15) with the knowledge that to any solution in which w, w' do not vanish corresponds essentially one periodic solution for the x_k, y_k of parameter σ/ν_0. We assume that *in general** $c_6 \neq 0$, $c_7 \neq 0$ (except for $\nu_0 = 2$, when by (5.28) we always have $c_7 = 0$, and in general $c_6 \neq 0$), and shall treat only this case in detail. Then from (7.12) and the first of (7.15) we have

$$w \frac{\partial L}{\partial w} = w' \frac{\partial L}{\partial w'} = \pm \left(v^{\nu_0} \frac{\partial L}{\partial w} \frac{\partial L}{\partial w'} \right)^{\frac{1}{2}};$$

substituting for w' in the second of (7.15) and cancelling a factor $v^{\frac{1}{2}\nu_0}$ this becomes

$$\frac{\partial L}{\partial v} \pm \nu_0 v^{\frac{1}{2}\nu_0 - 1} \left(\frac{\partial L}{\partial w} \frac{\partial L}{\partial w'} \right)^{\frac{1}{2}} = 0.$$

* In § 8 below an actual case is worked through in which this is so; for the complete justification of the assumption, see § 13, p. 214, below. The circumstances are similar for other constants which we shall assume later to be in general non-zero.

Substitute here and in (7.14) the series (7.6) for L, putting $U_2 = 0$ and writing $v^{\frac{1}{2}} = r$, we obtain

$$\left.\begin{aligned} w\,(c_6 + \ldots) \mp r^{\nu_0}(c_6 c_7 + \ldots)^{\frac{1}{2}} &= 0 \\ w'\,(c_7 + \ldots) \mp r^{\nu_0}(c_6 c_7 + \ldots)^{\frac{1}{2}} &= 0 \\ c'_3\,(\sigma - \nu) + c_4 Z_2 + c_{10} r^2 + c_{11} w + c_{12} w' + \ldots \pm \nu_0 r^{\nu_0 - 2}(c_6 c_7 + \ldots)^{\frac{1}{2}} &= 0 \\ c_2\,(\sigma - \nu) - a Z_2 + c_4 r^2 + c_{13} w + c_{14} w' + \ldots &= 0 \end{aligned}\right\} . \quad (7.21)$$

Under the hypothesis $c_6 c_7 \neq 0$ the left-hand sides become power series in $(\sigma - \nu)$, r, w, w', Z_2 on expanding the square roots.

CASE II*: $\nu_0 > 4$.—The equations (7.21) are soluble for w, w', Z_2, $(\sigma - \nu)$ as power series in r provided

$$ac'_3 + c_2 c_4 \neq 0, \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (7.22)$$

which we assume to be true in general.† Corresponding to the ambiguity of sign in (7.21) we have therefore *two* continuous families of equilibrium solutions for v, w, w', Z_2, and hence two families of periodic solutions for the x_k, y_k, the family-parameter being r—Families II, III, say. The two families have in common the solution for which $r = w = w' = Z_2 = \sigma - \nu = 0$, viz., the generating solution, which belongs also to Family I. The parameter of a solution for which r, w, w' are not zero is σ/ν_0, so in the neighbourhood of the generating solution the period for Families II, III is approximately ν_0 times as great as for Family I. We regard Families II, III as *branching* from Family I at the generating solution; they are evidently analogous to POINCARÉ's "solutions du deuxième genre."‡

We may solve (7.21) for w, w', r, Z_2 in powers of $(\sigma - \nu)^{\frac{1}{2}}$ provided

$$c_4^2 + ac_{10} \neq 0,$$

which we assume to be true in general. The leading terms in the solution are

$$\left.\begin{aligned} Z_2 &= \frac{c_2 c_{10} - c'_3 c_4}{c_4^2 + ac_{10}}(\sigma - \nu) + \ldots, \qquad r = \left(-\frac{ac_3' + c_2 c_4}{c_4^2 + ac_{10}}\right)^{\frac{1}{2}}(\sigma - \nu)^{\frac{1}{2}} + \ldots \\ w &= \pm\left(\frac{c_7}{c_6}\right)^{\frac{1}{2}}\left(-\frac{ac'_3 + c_2 c_4}{c_4^2 + ac_{10}}\right)^{\frac{1}{2}\nu_0}(\sigma - \nu)^{\frac{1}{2}\nu_0} + \ldots, \\ w' &= \pm\left(\frac{c_6}{c_7}\right)^{\frac{1}{2}}\left(-\frac{ac'_3 + c_2 c_4}{c_4^2 + ac_{10}}\right)^{\frac{1}{2}\nu_0}(\sigma - \nu)^{\frac{1}{2}\nu_0} + \ldots \end{aligned}\right\} . \quad (7.22)$$

Hence from (7.18) the solution for Z_1, U_1 is for each family of the form

$$Z_1 = (\sigma - \nu)^{\frac{1}{2}} P, \qquad U_1 = (\sigma - \nu)^{\frac{1}{2}} Q,$$

where P, Q are power series in $(\sigma - \nu)^{\frac{1}{2}}$. Thus in the Laurent series in $e^{\sigma t/\nu_0}$ which specify the solution for the x_k, y_k the coefficients are in general power series in $(\sigma - \nu)^{\frac{1}{2}}$.

* Case I is that in which λ, ν are incommensurable (§ 6).

† See footnote on p. 167.

‡ "Méth. Nouv.," Chap. XXVIII.

Case III: $\nu_0 = 4$.—The results are the same as for Case II, except as regards the solution (7.22), where c_{10} must be replaced by $c_{10} \pm 4(c_6c_7)^{\frac{1}{2}}$.

Case IV: $\nu_0 = 3$.—If we write $\pm r = r'$ the ambiguity of sign disappears from (7.21), and since $c_2c_6c_7 \neq 0$ these are soluble for $w, w', r', (\sigma - \nu)$ as power series in Z_2. Hence, there is, in addition to Family I, *one* family of periodic solutions, the generating solution being, as before, common to the two families. For the second family the period is approximately three times as great as for the first family in this neighbourhood.

If $a \neq 0$ we may solve (7.21) for w, w', Z_2, r' as power series in $(\sigma - \nu)$:

$$\left.\begin{aligned} r' &= -\frac{(c'_3 + c_2c_4/a)(\sigma-\nu)}{3(c_6c_7)^{\frac{1}{2}}} + \ldots, \qquad Z_2 = \frac{c_2}{a}(\sigma - \nu) + \ldots, \\ w &= \frac{-(c'_3 + c_2c_4/a)^3(\sigma-\nu)^3}{27c_6^2c_7} + \ldots, \qquad w' = \frac{-(c'_3 + c_2c_4/a)^3(\sigma-\nu)^3}{27c_6c_7^2} + \ldots \end{aligned}\right\}; \tag{7.23}$$

but if $a = 0$ the solution is in fractional powers of $(\sigma - \nu)$. From (7.18) we then have in general

$$Z_1 = (\sigma - \nu)P, \qquad U_1 = (\sigma - \nu)Q,$$

where P, Q are power series in $(\sigma - \nu)$. Thus in this case, in the Laurent series specifying the solution for the x_k, y_k the coefficients are in general power series in $(\sigma - \nu)$.

Case V: $\nu_0 = 2$.—We now always have $c_7 = 0$, and in general $c_6 \neq 0$, and the treatment of the conditions (7.12), (7.14), (7.15) is a little different. Supposing $c_6 \neq 0$, so that $\partial L/\partial w$ does not vanish in the neighbourhood of the origin, (7.15) give

$$w = w'\frac{\partial L}{\partial w'}\Big/\frac{\partial L}{\partial w}, \qquad v = -\tfrac{1}{2}w'\frac{\partial L}{\partial v}\Big/\frac{\partial L}{\partial w}, \tag{7.24}$$

the right-hand sides being expansible as power series in v, w, w', Z_2 $(\sigma - \nu)$. The condition (7.12) becomes on substitution of these values

$$w'^2\left\{4\frac{\partial L}{\partial w}\frac{\partial L}{\partial w'} - \left(\frac{\partial L}{\partial v}\right)^2\right\} = 0,$$

We reject the solution $w' = 0$ which leads to $v = w = 0$, and so to Family I as before, and have then

$$4\frac{\partial L}{\partial w}\frac{\partial L}{\partial w'} - \left(\frac{\partial L}{\partial v}\right)^2 = 0. \tag{7.25}$$

The leading terms in (7.24), (7.25), (7.14) are

$$\left.\begin{aligned} w - w'\{c_9(\sigma-\nu) + c_{12}v + c_{14}Z_2 + c_{18}w' + c_{19}w + \ldots\}/c_6 &= 0 \\ v + w'\{c'_3(\sigma-\nu) + c_4Z_2 + c_{10}v + c_{11}w + c_{12}w' + \ldots\}/2c_6 &= 0 \\ 4(c_6 + \ldots)\{c_9(\sigma-\nu) + c_{12}v + c_{14}Z_2 + c_{18}w' + c_{19}w + \ldots\} & \\ - \{c'_3(\sigma-\nu) + c_4Z_2 + c_{10}\nu + c_{11}w + c_{12}w' + \ldots\}^2 &= 0 \\ c_2(\sigma-\nu) - aZ_2 + c_4v + c_{13}w + c_{14}w' + \ldots &= 0 \end{aligned}\right\},$$

and if $c_{14}^2 + ac_{18} \neq 0$ we may solve for v, w, w', Z_2 in powers of $(\sigma - \nu)$:

$$\left.\begin{aligned} Z_2 &= \frac{c_2 c_{18} - c_9 c_{14}}{c_{14}^2 + ac_{18}}(\sigma - \nu) + \dots, \quad & w' &= -\frac{ac_9 + c_2 c_{14}}{c_{14}^2 + ac_{18}}(\sigma - \nu) + \dots \\ v &= (\text{constant}) \times (\sigma - \nu)^2 + \dots, \quad & w &= (\text{constant}) \times (\sigma - \nu)^3 + \dots \end{aligned}\right\} . \quad (7.26)$$

From (7.18) we then have

$$Z_1 = (\sigma - \nu)^{3/2} P, \quad U_1 = (\sigma - \nu)^{1/2} Q,$$

where P, Q are power series in $(\sigma - \nu)$.

We thus obtain, in addition to Family I, *one* family of periodic solutions for the x_k, y_k, which are Laurent series in $e^{\frac{1}{2}\sigma t}$ with coefficients in general power series in $(\sigma - \nu)^{\frac{1}{2}}$. As before, this second family branches from the first family at the generating solution.

Other Cases.—In the preceding investigation we have supposed that c_6, c_7, $ac'_3 + c_2 c_4, \dots$ do not vanish. Should any of these constants vanish the solution of the conditions (7.10) may be of higher multiplicity, but evidently cannot be of lower multiplicity; *exceptional cases can never give fewer families of periodic solutions than we have found.* Take, for example, the very special case in which K does not involve either of the arguments w, w'; the conditions (7.10) are then

$$Z_1 \frac{\partial L}{\partial v} = U_1 \frac{\partial L}{\partial v} = \frac{\partial L}{\partial Z_2} = 0$$

so that either

$$Z_1 = U_1 = \frac{\partial L}{\partial Z_2} = 0 \quad \text{or} \quad \frac{\partial L}{\partial v} = \frac{\partial L}{\partial Z_2} = 0,$$

where, as before, we may put $U_2 = 0$. The former conditions lead to Family I as before. In general the latter give Z_2 and v as power series $(\sigma - \nu)$, so in (7.4) only Z_2, U_2 and the product $Z_1 U_1$ are determined. We obtain thus a *doubly-infinite* family of periodic solutions, instead of two singly-infinite families, branching from Family I. In § 2 we have seen that this is in general the case when the equations (5.1) form a "soluble" system.

(ii) *The Exponents of the Periodic Solutions.*—We have seen (§ 5, p. 156) that the exponents of a solution for the x_k, y_k are the same as those of the corresponding solution for the z_k, u_k. Moreover, since the Z_k, U_k are linear functions of the z_k, u_k with coefficients which are periodic functions of t of parameter σ/ν_0, the exponents of a solution *of Family II or III* will be the same as those of the corresponding solution for the Z_k, U_k. We use, therefore, for Family I the co-ordinates z_k, u_k, for Families II, III the co-ordinates Z_k, U_k.

Family I.—A solution of this family is of the form

$$z_1 = u_1 = u_2 = 0, \quad z_2 = z_2^0. \quad \dots \dots \dots \quad (7.27)$$

To find its exponents we form the variational equations by substituting in (5.21) $z_2 = z_2^0 + \zeta_2$, and expanding to the first order in z_1, u_1, ζ_2, u_2. We have

$$\frac{dz_1}{dt} = \frac{\partial K}{\partial u_1} = z_1 \frac{\partial K}{\partial v} + \nu_0 u_1^{\nu_0 - 1} (\gamma e^{\sigma t})^{\lambda_0} \frac{\partial K}{\partial w'}.$$

$$\frac{du_1}{dt} = -\frac{\partial K}{\partial z_1} = -u_1 \frac{\partial K}{\partial v} - \nu_0 z_1^{\nu_0 - 1} (\gamma e^{\sigma t})^{-\lambda_0} \frac{\partial K}{\partial w},$$

and to the first order

$$\frac{d\zeta_2}{dt} = z_1 \left(\frac{\partial^2 K}{\partial u_2 \partial z_1}\right)_0 + u_1 \left(\frac{\partial^2 K}{\partial u_2 \partial u_1}\right)_0 + \zeta_2 \left(\frac{\partial^2 K}{\partial u_2 \partial z_2}\right)_0 + u_2 \left(\frac{\partial^2 K}{\partial u_2^2}\right)_0$$

$$\frac{du_2}{dt} = -z_1 \left(\frac{\partial^2 K}{\partial z_2 \partial z_1}\right)_0 - u_1 \left(\frac{\partial^2 K}{\partial z_2 \partial u_1}\right)_0 - \zeta_2 \left(\frac{\partial^2 K}{\partial z_2^2}\right)_0 - u_2 \left(\frac{\partial^2 K}{\partial z_2 \partial u_2}\right)_0,$$

where in the derivatives of K we must put $z_1 = u_1 = u_2 = 0$, $z_2 = z_2^0$. Thus

$$\left(\frac{\partial^2 K}{\partial u_2 \partial z_1}\right)_0 = \left\{u_1 \frac{\partial^2 K}{\partial u_2 \partial v} + \nu_0 z_1^{\nu_0 - 1} \frac{\partial^2 K}{\partial u_2 \partial w} (\gamma e^{\sigma t})^{-\lambda_0}\right\}_0,$$

$$= 0 \text{ since } \nu_0 \geq 2,$$

and so on, so the second pair of equations have the same form as (6.9) and have a similar solution. This gives that part of the solution of the variational equations which belongs to the two *zero* exponents of the generating solution (7.27). If $\nu_0 > 2$, *i.e.*, in Cases II, III, IV, the first pair of equations are to the first order

$$\frac{dz_1}{dt} = mz_1, \qquad \frac{du_1}{dt} = -mu_1, \qquad m = \left(\frac{\partial K}{\partial v}\right)_0,$$

having the solution $z_1 = \alpha_1 e^{mt}$, $u_1 = \beta_1 e^{-mt}$; hence the non-zero exponents of the solution (7.27) are $\pm m$. But if $\nu_0 = 2$ the first pair are to the first order

$$\frac{dz_1}{dt} = mz_1 + 2\gamma e^{\sigma t} \left(\frac{\partial K}{\partial w'}\right)_0 u_1, \qquad \frac{du_1}{dt} = -mu_1 - 2(\gamma e^{\sigma t})^{-1} \left(\frac{\partial K}{\partial w}\right)_0 z_1,$$

having the solution

$$z_1 = \alpha_1 . 2\gamma \left(\frac{\partial K}{\partial w'}\right)_0 e^{(\mu + \frac{1}{2}\sigma)t} + \beta_1 . 2\gamma \left(\frac{\partial K}{\partial w'}\right)_0 e^{(-\mu + \frac{1}{2}\sigma)t},$$

$$u_1 = \alpha_1 (\tfrac{1}{2}\sigma + \mu - m) e^{(\mu - \frac{1}{2}\sigma)t} + \beta_1 (\tfrac{1}{2}\sigma - \mu - m) e^{(-\mu - \frac{1}{2}\sigma)t},$$

where α_1, β_1 are arbitrary constants and

$$\mu = \left\{(m - \tfrac{1}{2}\sigma)^2 - 4\left(\frac{\partial K}{\partial w}\frac{\partial K}{\partial w'}\right)_0\right\}^{\frac{1}{2}}.$$

Here

$$\frac{\partial K}{\partial w} = \frac{\partial L}{\partial w}, \quad \frac{\partial K}{\partial w'} = \frac{\partial L}{\partial w'}, \quad m - \tfrac{1}{2}\sigma = \left\{\frac{\partial}{\partial v}(K - \tfrac{1}{2}\sigma v)\right\}_0 = \left(\frac{\partial L}{\partial v}\right)_0;$$

hence for the solution (7.24) the non-zero exponents are now

$$\pm\left[\tfrac{1}{2}\sigma + \left\{\left(\frac{\partial L}{\partial v}\right)_0^2 - 4\left(\frac{\partial L}{\partial w}\frac{\partial L}{\partial w'}\right)_0\right\}^{\frac{1}{2}}\right].$$

Inserting the series (7.6) we obtain as far as the leading terms

$$\mu = \pm\left[\tfrac{1}{2}\sigma + \{(c'_3(\sigma-\nu) + c_4 Z_2 + \ldots)^2 - 4(c_6 c_9(\sigma-\nu) + c_6 c_{14} Z_2 + \ldots)\}^{\frac{1}{2}}\right],$$

or

$$\mu = \pm\left[\tfrac{1}{2}\sigma + \{-4c_6(c_9 a + c_{14} c_2) Z_2/c_2 + \ldots\}^{\frac{1}{2}}\right], \quad \ldots \quad (7.28)$$

on inserting the value of $(\sigma - \nu)$ in terms of Z_2 appropriate to the solution (7.27).

Families II, III.—To a solution of either of these families corresponds an equilibrium solution for the Z_k, U_k, say $Z_k = Z_k^0$, $U_k = U_k^0$. We form its variational equations by substituting in (7.2)

$$Z_k = Z_k^0 + \zeta_k, \qquad U_k = U_k^0 + \omega_k,$$

and expanding to the first order in the ζ_k, ω_k, giving

$$\left.\begin{aligned}
\frac{d\zeta_1}{dt} &= \zeta_1\frac{\partial^2 L}{\partial Z_1\,\partial U_1} + \omega_1\frac{\partial^2 L}{\partial U_1^2} + \zeta_2\frac{\partial^2 L}{\partial Z_2\,\partial U_1} + \omega_2\frac{\partial^2 L}{\partial U_2\,\partial U_1}\\
\frac{d\omega_1}{dt} &= -\zeta_1\frac{\partial^2 L}{\partial Z_1^2} - \omega_1\frac{\partial^2 L}{\partial U_1\,\partial Z_1} - \zeta_2\frac{\partial^2 L}{\partial Z_2\,\partial Z_1} - \omega_2\frac{\partial^2 L}{\partial U_2\,\partial Z_1}\\
\frac{d\zeta_2}{dt} &= \zeta_1\frac{\partial^2 L}{\partial Z_1\,\partial U_2} + \omega_1\frac{\partial^2 L}{\partial U_1\,\partial U_2} + \zeta_2\frac{\partial^2 L}{\partial Z_2\,\partial U_2} + \omega_2\frac{\partial^2 L}{\partial U_2^2}\\
\frac{d\omega_2}{dt} &= -\zeta_1\frac{\partial^2 L}{\partial Z_1\,\partial Z_2} - \omega_1\frac{\partial^2 L}{\partial U_1\,\partial Z_2} - \zeta_2\frac{\partial^2 L}{\partial Z_2^2} - \omega_2\frac{\partial^2 L}{\partial U_2\,\partial Z_2}
\end{aligned}\right\},$$

where in the derivatives of L we substitute for the Z_k, U_k their equilibrium values. These are a set of linear equations with constant coefficients, and the characteristic equation is

$$\begin{vmatrix}
\frac{\partial^2 L}{\partial Z_1\,\partial U_1} - \mu, & \frac{\partial^2 L}{\partial U_1^2}, & \frac{\partial^2 L}{\partial Z_2\,\partial U_1}, & \frac{\partial^2 L}{\partial U_2\,\partial U_1}\\
\frac{\partial^2 L}{\partial Z_1^2}, & \frac{\partial^2 L}{\partial Z_1\,\partial U_1} + \mu, & \frac{\partial^2 L}{\partial Z_2\,\partial Z_1}, & \frac{\partial^2 L}{\partial U_2\,\partial Z_1}\\
\frac{\partial^2 L}{\partial Z_1\,\partial U_2}, & \frac{\partial^2 L}{\partial U_1\,\partial U_2}, & \frac{\partial^2 L}{\partial Z_2\,\partial U_2} - \mu, & \frac{\partial^2 L}{\partial U_2^2}\\
\frac{\partial^2 L}{\partial Z_1\,\partial Z_2}, & \frac{\partial^2 L}{\partial U_1\,\partial Z_2}, & \frac{\partial^2 L}{\partial Z_2^2}, & \frac{\partial^2 L}{\partial U_2\,\partial Z_2} + \mu
\end{vmatrix} = 0. \quad (7.29)$$

Now from (7.8) we have identically

$$\frac{\partial L}{\partial U_2} = \left(\frac{\partial L}{\partial Z_1}\frac{\partial N}{\partial U_1} - \frac{\partial L}{\partial U_1}\frac{\partial N}{\partial Z_1} + \frac{\partial L}{\partial Z_2}\frac{\partial N}{\partial U_2}\right)\Big/\frac{\partial N}{\partial Z_2},$$

where $\partial N/\partial Z_2$ does not vanish in the neighbourhood of the origin. Hence, since the first derivatives of L all vanish at the equilibrium solution, we have at the equilibrium solution

$$\frac{\partial^2 L}{\partial Z_1 \partial U_2} = \left(\frac{\partial^2 L}{\partial Z_1^2}\frac{\partial N}{\partial U_1} - \frac{\partial^2 L}{\partial Z_1 \partial U_1}\frac{\partial N}{\partial Z_1} + \frac{\partial^2 L}{\partial Z_1 \partial Z_2}\frac{\partial N}{\partial U_2}\right)\Big/\frac{\partial N}{\partial Z_2},$$

with similar expressions for

$$\frac{\partial^2 L}{\partial U_1 \partial U_2}, \qquad \frac{\partial^2 L}{\partial Z_2 \partial U_2}, \qquad \frac{\partial^2 L}{\partial U_2^2}.$$

Inserting these values in (7.29) it is found that two of the roots μ are zero and the other two are given by

$$\left(\frac{\partial N}{\partial Z_2}\right)^2 \mu^2 = \begin{vmatrix} \frac{\partial^2 L}{\partial Z_1^2}, & \frac{\partial^2 L}{\partial Z_1 \partial U_1}, & \frac{\partial^2 L}{\partial Z_1 \partial Z_2}, & \frac{\partial N}{\partial Z_1} \\ \frac{\partial^2 L}{\partial Z_1 \partial U_1}, & \frac{\partial^2 L}{\partial U_1^2}, & \frac{\partial^2 L}{\partial U_1 \partial Z_2}, & \frac{\partial N}{\partial U_1} \\ \frac{\partial^2 L}{\partial Z_1 \partial Z_2}, & \frac{\partial^2 L}{\partial U_1 \partial Z_2}, & \frac{\partial^2 L}{\partial Z_2^2}, & \frac{\partial N}{\partial Z_2} \\ \frac{\partial N}{\partial Z_1}, & \frac{\partial N}{\partial U_1}, & \frac{\partial N}{\partial Z_2}, & 0 \end{vmatrix} \quad \ldots \ldots \quad (7.30)$$

These four roots are the exponents of the solution under consideration. By means of the relations

$$\frac{\partial}{\partial Z_1} = U_1\left(\frac{\partial}{\partial v} + \frac{\nu_0 w}{v}\frac{\partial}{\partial w}\right), \qquad \frac{\partial}{\partial U_1} = Z_1\left(\frac{\partial}{\partial v} + \frac{\nu_0 w'}{v}\frac{\partial}{\partial w'}\right),$$

$$\frac{\partial^2}{\partial Z_1^2} = U_1^2\left(\frac{\partial^2}{\partial v^2} + \frac{2\nu_0 w}{v}\frac{\partial^2}{\partial v\,\partial w} + \frac{\nu_0^2 w^2}{v^2}\frac{\partial^2}{\partial w^2} + \frac{\nu_0^2 - \nu_0}{v^2} w \frac{\partial}{\partial w}\right),$$

$$\frac{\partial^2}{\partial Z_1 \partial U_1} = \frac{\partial}{\partial v} + v\frac{\partial^2}{\partial v^2} + \nu_0 w\frac{\partial^2}{\partial v\,\partial w} + \nu_0 w'\frac{\partial^2}{\partial v\,\partial w'} + \frac{\nu_0^2 w w'}{v}\frac{\partial^2}{\partial w\,\partial w'},$$

$$\frac{\partial^2}{\partial U_1^2} = Z_1^2\left(\frac{\partial^2}{\partial v^2} + \frac{2\nu_0 w'}{v}\frac{\partial^2}{\partial v\,\partial w'} + \frac{\nu_0^2 w'^2}{v^2}\frac{\partial^2}{\partial w'^2} + \frac{\nu_0^2 - \nu_0}{v^2} w'\frac{\partial}{\partial w'}\right),$$

we may express μ^2 in terms of derivatives with respect to v, w, w', Z_2; then, substituting the values of v, w, w', Z_2, U_2 in terms of $(\sigma - \nu)$ which are appropriate to the particular solution, we obtain μ^2 in terms of $(\sigma - \nu)$—in general as a power series in $(\sigma - \nu)^{\frac{1}{3}}$. The results are, as far as the leading term of the series:

CASE II: $\nu_0 > 4$.

$$\mu^2 = \pm\frac{2\nu_0^2\,(ac'^2_3 + 2c_2c'_3c_4 - c_2^2c_{10})\,(c_6c_7)^{\frac{1}{2}}}{c_2^2}\left(-\frac{ac'_3 + c_2c_4}{c_4^2 + ac_{10}}\right)^{\frac{1}{2}\nu_0}(\sigma - \nu)^{\frac{1}{2}\nu_0} + \ldots; \quad . \quad . \quad (7.31)$$

the upper and lower signs refer to Families II, III, respectively.

CASE III: $\nu_0 = 4$.—We obtain the leading term in the series for μ^2 if in (7.31) we replace c_{10} by $c_{10} \pm 4\,(c_6c_7)^{\frac{1}{2}}$; as before, the upper and lower signs refer to Families II, III, respectively.

CASE IV: $\nu_0 = 3$.

$$\mu^2 = -3(c'_3 + c_2c_4/a)^2(\sigma - \nu)^2 + \dots \quad (7.32)$$

CASE V: $\nu_0 = 2$.

$$\mu^2 = \frac{8c_6(c_2^2c_{18} - ac_9^2 - 2c_2c_9c_{14})(ac_9 + c_2c_{14})}{c_2^2(c_{14}^2 + ac_{18})}(\sigma - \nu) + \dots \quad (7.33)$$

(iii) *Reality of the Periodic Solutions.*—Suppose now that F is a real function of the x_k, y_k, and that the generating solution (5.2) is real and of real period, so that ν is pure-imaginary; then λ, being commensurable with ν, is also pure-imaginary. We consider throughout only those solutions for which σ is pure-imaginary, t is real and $|\gamma| = 1$.

CASES II, III, IV: $\lambda \neq \frac{1}{2}\nu$.—The x_k, y_k are real functions of the coordinates z'_1, u'_1, z_2, u_2 given by (5.32) or (5.33). Writing $\overline{f}$ for the function conjugate to f we have from these relations $\overline{z_1} = \mp \iota u_1$,* and hence from (7.1) $\overline{Z_1} = \mp \iota U_1$. Thus

$$\overline{w} = \overline{Z_1}^{\nu_0} = (\mp \iota U_1)^{\nu_0} = (\mp \iota)^{\nu_0} w',$$

so that $s, = w + (\mp \iota)^{\nu_0} w'$, and $s', = \iota w - \iota(\mp \iota)^{\nu_0} w'$, are real functions of z_1', u_1' z_2, u_2. Also $v = z_1u_1 = \pm \frac{1}{2}\iota(z'^2_1 + u'^2_1), = \pm \iota q$, say. We may change from the arguments v, w, w' to q, s, s' which are real with z'_1, u'_1, and L must become a power series in q, s, s', Z_2, U_2, $\iota(\sigma - \nu)$ with real coefficients, say,

$$L(v, w, w', Z_2, U_2, \sigma - \nu) \equiv H(q, s, s', Z_2, U_2, \iota\sigma - \iota\nu).$$

For Family I, $q = s = s' = U_2 = 0$ while Z_2, $\iota(\sigma - \nu)$ are connected by the relation

$$\frac{\partial H}{\partial Z_2} = 0;$$

the left-hand side is a power series in Z_2, $\iota(\sigma - \nu)$ with real coefficients, and always yields $\iota(\sigma - \nu)$ as a real power series in Z_2. Hence those solutions of the family for which Z_2, t are real and $|\gamma| = 1$ are real, and have their periods $2\pi\iota/\sigma$ real. The non-zero exponents of such a solution are

$$\pm\left(\frac{\partial K}{\partial v}\right)_0 = \pm\iota\left(\frac{\partial H}{\partial q}\right)_0,$$

and are thus pure-imaginary.

Now consider Families II, III, which are obtained by solving (7.12), (7.14), (7.15). When expressed in terms of q, s, s' instead of v, w, w' these conditions are equivalent to

$$\left.\begin{aligned} s^2 + s'^2 - 4q^{\nu_0} &= 0 \\ s'\frac{\partial H}{\partial s} - s\frac{\partial H}{\partial s'} &= 0 \\ s\frac{\partial H}{\partial q} + 2\nu_0 q^{\nu_0-1}\frac{\partial H}{\partial s} &= 0 \\ \frac{\partial H}{\partial Z_2} &= 0 \end{aligned}\right\},$$

* Here and below the upper sign refers to Case A, the lower to Case B.

the left-hand sides being power series in q, s, s', Z_2, U_2, $\iota(\sigma - \nu)$ with real coefficients (where we put $U_2 = 0$). Corresponding to the two solutions previously found for w, w', Z_2, $(\sigma - \nu)$ in powers of $v^{\frac{1}{2}}$, we obtain two solutions for s, s', Z_2, $\iota(\sigma - \nu)$ in powers of $q^{\frac{1}{2}}$, and these are easily seen to be real when q is positive. Moreover, to a real solution for s, s' corresponds a real solution for z'_1, u'_1, for when s, s' are real we have

$$\overline{w} = (\mp \iota)^{\nu_0} w',$$

and so

$$\overline{w^{1/\nu_0}} = (\mp \iota)\, w'^{\,1/\nu_0},$$

$$\overline{z}_1 e^{\sigma\lambda_0 t/\nu_0} = \mp\, \iota u_1 e^{\sigma\lambda_0 t/\nu_0},$$

$$\overline{z_1'} - \iota\overline{u_1'} = z_1' - \iota u_1',$$

so that z'_1, u'_1 are real. Hence those solutions of Families II, III for which q is positive, t is real and $|\gamma| = 1$ are real, and have real periods.

Considering now the exponents of solutions of Families II, III, the expression (7.30) for μ^2 is symmetrical with respect to Z_1, U_1, and dimensionally is of degree -4 in them. Hence since

$$\overline{Z}_1 = \mp\, \iota U_1, \qquad \overline{U}_1 = \mp\, \iota Z_1,$$

we have $\overline{\mu^2} = \mu^2$, *i.e.*, μ^2 is a real function of z'_1, u'_1, Z_2, U_2, $\iota(\sigma - \nu)$. Its sign when $|\sigma - \nu|$ is small can be inferred from that of the leading term in the series for μ^2 as already given in (7.31), (7.32). Thus—

when $\nu_0 > 4$, μ^2 is positive for Family II and negative for Family III, or *vice versa*;

when $\nu_0 = 4$, μ^2 may be positive for both families, negative for both, or positive for one and negative for the other, according to the values of c_2, c_3, ...;

when $\nu_0 = 3$, μ^2 is positive, for in (7.32) $(\sigma - \nu)$ is pure-imaginary and $c'_3 + c_2 c_4/a$ is real.* Of course the non-zero exponents $\pm\,\mu$ of a solution are real when μ^2 is positive and pure-imaginary when μ^2 is negative.

Case V: $\lambda = \frac{1}{2}\nu$.—From (5.34) we see that when σ, ν are pure-imaginary, t real and $|\gamma| = 1$, the x_k, y_k are real functions of Z_1, U_1, Z_2, U_2; hence L is a power series in v, w, w', Z_2, U_2, $\iota(\sigma - \nu)$ with real coefficients.

For a solution of Family I we have $Z_1 = U_1 = U_2 = 0$, while $(\sigma - \nu)$ is given in terms of Z_2 by the relation

$$\left(\frac{\partial L}{\partial Z_2}\right)_{Z_1 = U_1 = U_2 = 0} = 0,$$

of which the left-hand side is a power series in Z_2, $\iota(\sigma - \nu)$ with real coefficients. The solution for $\iota(\sigma - \nu)$ in powers of Z_2 is therefore real, so those solutions of Family I for which Z_2 is real are real. In the expression (7.28) for their non-zero exponents the term $\frac{1}{2}\sigma$ is pure-imaginary, while the expression under the radical is a power series in Z_2 with real coefficients, which in the neighbourhood of the origin changes

* c'_3 is real because it is the coefficient of $(\sigma-\nu)\,v$ in L and therefore of $\iota(\sigma-\nu)\,q$ in H; a and c_2c_4 may similarly be shown to be real.

its sign with Z_2 provided $c_6(ac_9 + c_2c_{14}) \neq 0$, which is so in general. Hence, in general the non-zero exponents for real solutions of Family I are pure-imaginary on one side of the generating solution and complex on the other, and a solution for which the exponents are complex has their imaginary part equal to half its parameter.

Family II is obtained by solution of the conditions (7.12), (7.14), (7.15), of which the left-hand sides are power series in v, w, w', Z_2, $\iota(\sigma - \nu)$ with real coefficients. In general there is a unique real solution (7.26) for v, w, w', Z_2 in powers of $\iota(\sigma - \nu)$, but Z_1 and U_1 proceed in powers of $\{\iota(\sigma - \nu)\}^{\frac{1}{2}}$, and will be real for $\iota(\sigma - \nu) > 0$ or for $\iota(\sigma - \nu) < 0$, but not for both. Hence the real solutions of Family II are either those for which the period is real and greater or those for which it is real and less than twice the period of the generating solution.

The expression (7.30) gives μ^2 as a real function of Z_1, U_1, Z_2, $\iota(\sigma - \nu)$, so a real solution of Family II has its non-zero exponents either real or pure-imaginary.*

(iv) *Graphical Representation of Real Periodic Solutions having a Real Period.*—The relations between the real periodic solutions of the various families may be exhibited graphically by representing each such solution by a point whose abscissa is the appropriate value of $\iota(\sigma - \nu)$, and whose ordinate is the constant value of F belonging to the solution in question. [The reason for choosing F and $\iota(\sigma - \nu)$ as co-ordinates is that they are quantities which for any definite periodic solution have definite values not in any way dependent on the transformations (5.20), (7.4).] If F_0 is the value of F belonging to the generating solution we have in the neighbourhood of the generating solution

$$\begin{aligned} F - F_0 &= L - N, \\ &= -\nu\,(c_2Z_2 + d_2U_2 + c_3'v + \ldots) - \tfrac{1}{2}aZ_2^2 + c_4Z_2v + \ldots, \end{aligned}$$

from (7.6), (7.7). The relation between F and $\iota(\sigma - \nu)$ for Family I, II or III is then found by substituting $U_2 = 0$ and the values of Z_2, v, w, w' in terms of $(\sigma - \nu)$, as already found. For instance, for Family I

$$v = w = w' = 0, \quad aZ_2 = c_2(\sigma - \nu) + \ldots, \quad F - F_0 = -\nu c_2^2(\sigma - \nu)/a + \ldots, \qquad (7.34)$$

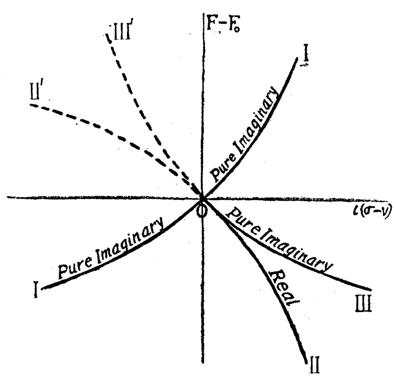
and for Families II, III, when $\nu_0 > 4$ (using (7.22))

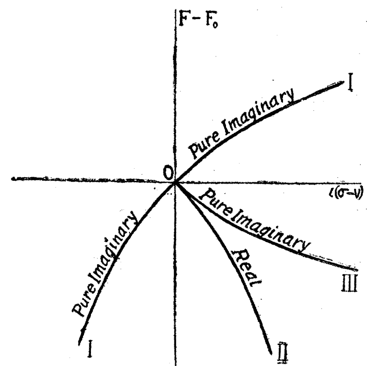
$$F - F_0 = -\nu\left(\frac{c_2^2c_{10} - ac_3'^2 - 2c_2c_3'c_4}{c_4^2 + ac_{10}}\right)(\sigma - \nu) + \ldots, \quad . \quad . \quad . \qquad (7.35)$$

the series on the right being the same for the two families as far as the coefficient of $(\sigma - \nu)^{\frac{1}{2}\nu_0 - 1}$. Since to a real solution corresponds a real value of F, we know from what has preceded that (7.34) gives F as a real function of $\iota(\sigma - \nu)$, and that (7.35) gives F real for some values of $\iota(\sigma - \nu)$ in the neighbourhood of the origin; when ν_0 is even (7.35) is a power series in $\iota(\sigma - \nu)$, and so must be real for $\iota(\sigma - \nu) > 0$ and $\iota(\sigma - \nu) < 0$,

* Of course, all solutions of the Family sufficiently near the generating solution have their exponents of the same nature; they may be real or pure-imaginary.

while when ν_0 is odd (7.35) is a power series in $\{\iota(\sigma - \nu)\}^{\frac{1}{2}}$ and $F - F_0$ is real *either* for $\iota(\sigma - \nu) > 0$ *or* for $\iota(\sigma - \nu) < 0$. We obtain thus a diagram of the sort shown in fig. 1 or 2. Here each point on the curve I O I corresponds to a solution of Family I, and each point on the curves II O, III O, II′ O, III′ O to a solution of Family II or III. The curves II O, III O touch at O. Now we know that of Families II, III, only those solutions are real for which $q\,(= \mp \iota v)$ is positive, and from (7.22) we see that q changes sign with





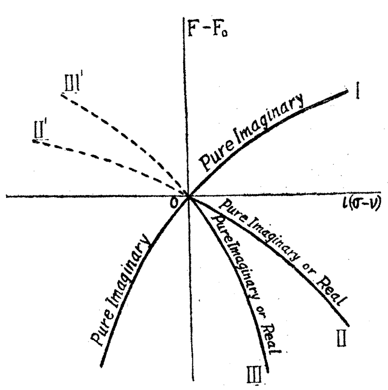
ν_0 even. ν_0 odd.

$\nu_0 > 4$, $a \neq 0$, $c_6 \neq 0$, $c_7 \neq 0$, $ac_3' + c_2c_4 \neq 0$.

FIG. 1. FIG. 2.

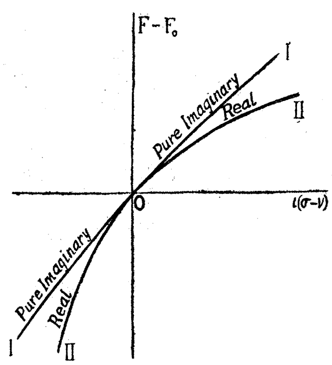
$\iota(\sigma - \nu)$; hence, in fig. 1, if the points of II O, III O correspond to real solutions, those of II′ O, III′ O will correspond to imaginary solutions, or *vice versa*. The branches II′ O, III′ O have been drawn dotted to indicate that their points correspond to imaginary solutions. Along each curve is written the nature of the non-zero exponents of the periodic solutions represented by its points.

The corresponding diagrams for the other cases are as follows :—



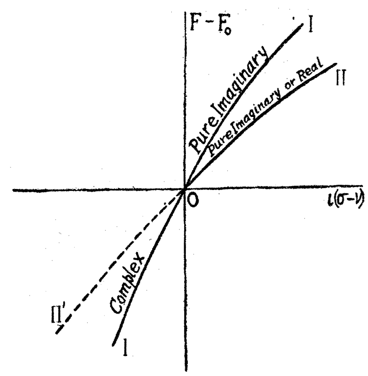
$\nu_0 = 4$, $a \neq 0$, $c_6 \neq 0$, $c_7 \neq 0$, $ac'_3 + c_2c_4 \neq 0$.

FIG. 3.



$\nu_0 = 3$, $a \neq 0$, $c_6 \neq 0$, $c_7 \neq 0$.

FIG. 4.



$\nu_0 = 2$, $a \neq 0$, $c_6 \neq 0$, $ac_9 + c_2c_{14} \neq 0$.

FIG. 5.

Case III: $\nu_0 = 4$.—Here (fig. 3) the curves representing Families II, III have distinct tangents at the origin, which may lie in the same quadrant or different quadrants, according to the values of $c_2, c'_3, c_4, \ldots$.

Case IV: $\nu_0 = 3$.—Here (fig. 4) the curves representing Families I, II touch at 0.

Case V: $\nu_0 = 2$.—Here (fig. 5) for Family II, $F - F_0$ is real both for $\iota(\sigma - \nu) < 0$ and for $\iota(\sigma - \nu) > 0$, but only a certain sign for $\iota(\sigma - \nu)$ leads to real solutions.

§ 8. *Example illustrating the Theory of* §§ 5, 7.

Take the equations

$$\frac{dx_k}{dt} = \frac{\partial F}{\partial y_k}, \qquad \frac{dy_k}{dt} = -\frac{\partial F}{\partial x_k} \qquad (k = 1, 2), \quad \ldots \ldots \quad (8.1)$$

where

$$F \equiv \tfrac{1}{2}\mu_1(x_1^2 + y_1^2) + \tfrac{1}{2}\mu_2(x_2^2 + y_2^2) + \tfrac{1}{8}a(x_1^2 + y_1^2)^2 + \tfrac{1}{4}b(x_1^2 + y_1^2)(x_2^2 + y_2^2)$$
$$+ \tfrac{1}{8}c(x_2^2 + y_2^2)^2 + 2^{-\frac{1}{2}(A_1 + A_2)}\{f(x_1 - \iota y_1)^{A_1}(x_2 - \iota y_2)^{A_2} + g(y_1 - \iota x_1)^{A_1}(y_2 - \iota x_2)^{A_2}\},$$

A_1, A_2 being positive integers greater than 2 and $\mu_1, \mu_2, a, b, c, f, g$ constants.* They possess the periodic solution

$$x_1 = y_1 = 0, \quad x_2 = (2v_2)^{\frac{1}{2}} \sin(\mu_2 + cv_2)t, \quad y_2 = (2v_2)^{\frac{1}{2}} \cos(\mu_2 + cv_2)t, \quad . . \quad (8.2)$$

where v_2 is an arbitrary constant.

The normalising contact transformation appropriate to this solution is

$$\left.\begin{aligned} \sqrt{2}x_1 &= \zeta_1 + \iota\omega_1, \qquad \sqrt{2}y_1 = \omega_1 + \iota\zeta_1 \\ \sqrt{2}x_2 &= -\sqrt{2}\omega_2 \cos\nu' t + \sqrt{2}\zeta_2 \sin\nu' t + 2\sqrt{v_2}\sin\nu' t \\ \sqrt{2}y_2 &= \sqrt{2}\omega_2 \sin\nu' t + \sqrt{2}\zeta_2 \cos\nu' t + 2\sqrt{v_2}\cos\nu' t \end{aligned}\right\}, \quad . . . \quad (8.3)$$

where $\nu' = \mu_2 + cv_2$, while the "ν" of the general theory of § 5 is given by $\nu = \iota\nu'$; under it the equations become

$$\frac{d\zeta_k}{dt} = \frac{\partial G}{\partial \omega_k}, \qquad \frac{d\omega_k}{dt} = -\frac{\partial G}{\partial \zeta_k} \qquad (k = 1, 2), \quad \ldots \ldots \quad (8.4)$$

with

$$G \equiv \iota(\mu_1 + bv_2)\zeta_1\omega_1 + cv_2\zeta_2^2 + \iota b(2v_2)^{\frac{1}{2}}\zeta_1\omega_1\zeta_2 + \tfrac{1}{2}c(2v_2)^{\frac{1}{2}}\zeta_2(\zeta_2^2 + \omega_2^2)$$
$$- \tfrac{1}{2}a\zeta_1^2\omega_1^2 + \tfrac{1}{2}\iota b\zeta_1\omega_1(\zeta_2^2 + \omega_2^2) + \tfrac{1}{8}c(\zeta_2^2 + \omega_2^2)^2$$
$$+ f\zeta_1^{A_1}e^{\iota A_2\nu' t}\left(-\frac{\omega_2 + \iota\zeta_2}{\sqrt{2}} - \iota\sqrt{v_2}\right)^{A_2} + g\omega_1^{A_1}e^{-\iota A_2\nu' t}\left(\frac{\zeta_2 + \iota\omega_2}{\sqrt{2}} + \sqrt{v_2}\right)^{A_2}.$$

The form of the second degree terms shows that the non-zero exponents of the solution (8.2) are $\pm\iota(\mu_1 + bv_2)$, $= \pm\lambda$ say.

* This is a "soluble" system; it reduces to that considered in § 2 when $A_1 = 2$, $A_2 = 1$, $a = b = c = 0$. F is a real function of the x_k, y_k provided μ_1, μ_2, a, b, c are real and $f = \iota^{A_1 + A_2}\bar{g}$.

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We take as generating solution that solution of the family (8.2) for which $\lambda/\nu = -A_2/A_1$, *i.e.*, $A_1(\mu_1 + bv_2) + A_2(\mu_2 + cv_2) = 0$ (giving the requisite value of the parameter v_2). If taking this value for v_2 we solve the equations (8.4), we find a solution of the form

$$\left.\begin{aligned} \zeta_1 &= \alpha_1 e^{\lambda t} + t\mathrm{P}_1, & \omega_1 &= \beta_1 e^{-\lambda t} + t\mathrm{Q}_1 \\ \zeta_2 &= \alpha_2 + t\mathrm{P}_2, & \omega_2 &= \beta_2 + t\mathrm{Q}_2 \end{aligned}\right\},$$

where the P_k, Q_k are power series in $\alpha_1 e^{\lambda t}$, $\beta_1 e^{-\lambda t}$, α_2, β_2, t. The transformation to the co-ordinates z_k, u_k is to be obtained by substituting this in (8.3), and then replacing α_k, β_k, $e^{\nu t}$, t by z_k, u_k, $e^{\sigma t}$, 0, respectively. This gives

$$\left.\begin{aligned} \sqrt{2}x_1 = z_1 + \iota u_1, \qquad \sqrt{2}y_1 &= u_1 + \iota z_1 \\ x_2 &= (2v_2)^{\frac{1}{2}} \sin \sigma' t + z_2 \sin \sigma' t - u_2 \cos \sigma' t \\ y_2 &= (2v_2)^{\frac{1}{2}} \cos \sigma' t + z_2 \cos \sigma' t + u_2 \sin \sigma' t \end{aligned}\right\}, \qquad (8.5)$$

where the σ of the general theory is put equal to $\iota\sigma'$.

The Hamiltonian function of the transformed system is $\mathrm{K}, = \mathrm{F} + \mathrm{M}$, where

$$\frac{\partial \mathrm{M}}{\partial z_1} = [z_1, t] = 0, \qquad \frac{\partial \mathrm{M}}{\partial u_1} = [u_1, t] = 0,$$

$$\frac{\partial \mathrm{M}}{\partial z_2} = [z_2, t] = -\sigma'(z_2 + (2v_2)^{\frac{1}{2}}), \qquad \frac{\partial \mathrm{M}}{\partial u_2} = [u_2, t] = -\sigma' u_2,$$

and so

$$\mathrm{M} = -\sigma'\{z_2(2v_2)^{\frac{1}{2}} + \tfrac{1}{2}z_2^2 + \tfrac{1}{2}u_2^2\};$$

while

$$\begin{aligned} \mathrm{F} = {} & \mu_2 v_2 + \tfrac{1}{2}cv_2^2 + (2v_2)^{\frac{1}{2}}\nu' z_2 + \tfrac{1}{2}\nu'(z_2^2 + u_2^2) + \lambda z_1 u_1 + cv_2 z_2^2 \\ & + \iota b(2v_2)^{\frac{1}{2}} z_1 u_1 z_2 + \tfrac{1}{2}c(2v_2)^{\frac{1}{2}} z_2(z_2^2 + u_2^2) - \tfrac{1}{2}az_1^2u_1^2 + \tfrac{1}{2}\iota bz_1u_1(z_2^2 + u_2^2) \\ & + \tfrac{1}{8}c(z_2^2 + u_2^2) + fz_1^{\mathrm{A}_1} e^{\iota\sigma'\mathrm{A}_2 t}\left(-\frac{u_2 + \iota z_2}{\sqrt{2}} - \iota\sqrt{v_2}\right)^{\mathrm{A}_2} + gu_1^{\mathrm{A}_1} e^{-\iota\sigma'\mathrm{A}_2 t}\left(\frac{z_2 + \iota u_2}{\sqrt{2}} + \sqrt{v_2}\right)^{\mathrm{A}_2}. \end{aligned}$$

Thus, writing in accordance with (5.26),

$$v = z_1 u_1, \qquad w = z_1^{\mathrm{A}_1} e^{\iota\sigma'\mathrm{A}_2 t}, \qquad w' = u_1^{\mathrm{A}_1} e^{-\iota\sigma'\mathrm{A}_2 t},$$

we have

$$\begin{aligned} \mathrm{K} \equiv {} & -(\sigma' - \nu')\{z_2(2v_2)^{\frac{1}{2}} + \tfrac{1}{2}z_2^2 + \tfrac{1}{2}u_2^2\} + \lambda v + cv_2 z_2^2 + \iota b(2v_2)^{\frac{1}{2}} vz_2 \\ & + \tfrac{1}{2}c(2v_2)^{\frac{1}{2}} z_2(z_2^2 + u_2^2) - \tfrac{1}{2}av^2 + \tfrac{1}{2}\iota bv(z_2^2 + u_2^2) + \tfrac{1}{8}c(z_2^2 + u_2^2)^2 \\ & + fw\left(-\frac{u_2 + \iota z_2}{\sqrt{2}} - \iota\sqrt{v_2}\right)^{\mathrm{A}_2} + gw'\left(\frac{z_2 + \iota u_2}{\sqrt{2}} + \sqrt{v_2}\right)^{\mathrm{A}_2}, \end{aligned}$$

where

$$\sigma' = -\iota\sigma, \qquad \nu' = -\iota\nu.$$

Hence

$$\mathrm{L} = \mathrm{K} - \frac{\sigma\lambda_0}{\nu_0} v = \mathrm{K} + \frac{\iota\sigma' \mathrm{A}_2}{\mathrm{A}_1} v$$

$$\equiv -(\sigma' - \nu')\left\{z_2 (2v_2)^{\frac{1}{2}} + \tfrac{1}{2}z_2^2 + \tfrac{1}{2}u_2^2 - \frac{\iota \mathrm{A}_2}{\mathrm{A}_1} v\right\} + cv_2 z_2^2 + \iota b (2v_2)^{\frac{1}{2}} v z_2$$

$$+ \tfrac{1}{2}c (2v_2)^{\frac{1}{2}} z_2 (z_2^2 + u_2^2) - \tfrac{1}{2}av^2 + \tfrac{1}{2}\iota bv (z_2^2 + u_2^2) + \tfrac{1}{8}c (z_2^2 + u_2^2)^2$$

$$+ fw\left(-\frac{u_2 + \iota z_2}{\sqrt{2}} - \iota\sqrt{v_2}\right)^{\mathrm{A}_2} + gw'\left(\frac{z_2 + \iota u_2}{\sqrt{2}} + \sqrt{v_2}\right)^{\mathrm{A}_2}.$$

Family I.—We obtain this family of periodic solutions from (8.5) by putting $z_1 = u_1 = 0$ and giving z_2, u_2 values satisfying

$$\left.\begin{aligned} \frac{\partial \mathrm{L}}{\partial z_2} &= -(\sigma' - \nu')(2v_2)^{\frac{1}{2}} - (\sigma' - \nu')z_2 + 2cv_2 z_2 + \tfrac{1}{2}c (2v_2)^{\frac{1}{2}} (z_2^2 + u_2^2) + c (2v_2)^{\frac{1}{2}} z_2^2 \\ &\qquad + \tfrac{1}{2}cz_2 (z_2^2 + u_2^2) = 0 \\ \frac{\partial \mathrm{L}}{\partial u_2} &= -(\sigma' - \nu')\, u_2 + c (2v_2)^{\frac{1}{2}} z_2 u_2 + \tfrac{1}{2}cu_2 (z_2^2 + u_2^2) = 0 \end{aligned}\right\},$$

i.e.,

$$\left.\begin{aligned} \{z_2 + (2v_2)^{\frac{1}{2}}\}\{-(\sigma' - \nu') + c (2v_2)^{\frac{1}{2}} z_2 + \tfrac{1}{2}c (z_2^2 + u_2^2)\} &= 0 \\ u_2 \{-(\sigma' - \nu') + c (2v_2)^{\frac{1}{2}} z_2 + \tfrac{1}{2}c (z_2^2 + u_2^2)\} &= 0 \end{aligned}\right\}.$$

We see that in accordance with the general theory the second equation is in the neighbourhood of $z_2 = u_2 = 0$ a consequence of the first, and the relation between z_2, u_2 is

$$\{z_2 + (2v_2)^{\frac{1}{2}}\}^2 + u_2^2 = 2(\sigma' - \nu')/c + 2v_2.$$

It is evident also that any two pairs of values of z_2, u_2 satisfying this, when substituted in (8.5), give solutions differing only in the value of a constant additive with t.

The periodic solution which arises from (8.5) by putting

$$z_1 = u_1 = u_2 = 0, \qquad z_2 + (2v_2)^{\frac{1}{2}} = \{2(\sigma' - \nu')/c + 2v_2\}^{\frac{1}{2}}$$

is easily seen to coincide with (8.2) if we there put $v_2 = (\sigma' - \mu_2)/c$. Thus Family I is represented by (8.2), with v_2 an arbitrary parameter.

Families II, III.—The relations (7.12), (7.15), (7.14) are

$$\left.\begin{aligned} &ww' = v^{\mathrm{A}_1} \\ &fw\left(-\frac{u_2 + \iota z_2}{\sqrt{2}} - \iota\sqrt{v_2}\right)^{\mathrm{A}_2} = gw'\left(\frac{z_2 + \iota u_2}{\sqrt{2}} + \sqrt{v_2}\right)^{\mathrm{A}_2} \\ &w'\left\{\frac{\iota \mathrm{A}_2}{\mathrm{A}_1}(\sigma' - \nu') + \iota b (2v_2)^{\frac{1}{2}} z_2 - av + \tfrac{1}{2}\iota b (z^2 + u_2^2)\right\} \\ &\qquad + \mathrm{A}_1 v^{\mathrm{A}_1 - 1} f\left(-\frac{u_2 + \iota z_2}{\sqrt{2}} - \iota\sqrt{v_2}\right)^{\mathrm{A}_2} = 0 \\ &\{z_2 + (2v_2)^{\frac{1}{2}}\}\{-(\sigma' - \nu') + c (2v_2)^{\frac{1}{2}} z_2 + \tfrac{1}{2}c (z_2^2 + u_2^2) + \iota bv\} \\ &\qquad - \frac{\iota fw \mathrm{A}_2}{\sqrt{2}}\left(-\frac{u_2 + \iota z_2}{\sqrt{2}} - \iota\sqrt{v_2}\right)^{\mathrm{A}_2 - 1} + \frac{gw' \mathrm{A}_2}{\sqrt{2}}\left(\frac{z_2 + \iota u_2}{\sqrt{2}} + \sqrt{v_2}\right)^{\mathrm{A}_2 - 1} = 0 \end{aligned}\right\}. \quad (8.6)$$

The first two give

$$\left.\begin{aligned} w &= \pm v^{\frac{1}{2}A_1}\left(\frac{g}{f}\right)^{\frac{1}{2}}\left\{-\frac{z_2+\iota u_2+(2v_2)^{\frac{1}{2}}}{u_2+\iota z_2+\iota(2v_2)^{\frac{1}{2}}}\right\}^{\frac{1}{2}A_2} \\ w' &= \pm v^{\frac{1}{2}A_1}\left(\frac{f}{g}\right)^{\frac{1}{2}}\left\{-\frac{u_2+\iota z_2+\iota(2v_2)^{\frac{1}{2}}}{z_2+\iota u_2+(2v_2)^{\frac{1}{2}}}\right\}^{\frac{1}{2}A_2} \end{aligned}\right\}, \quad \ldots \ldots \quad (8.7)$$

and the fourth becomes

$$\begin{aligned} &\{z_2+(2v_2)^{\frac{1}{2}}\}\{-(\sigma'-\nu')+c(2v_2)^{\frac{1}{2}}z_2+\tfrac{1}{2}c(z_2^2+u_2^2)+\iota bv\} \\ &\quad = \pm\iota(fg)^{\frac{1}{2}}A_2v^{\frac{1}{2}A_1}(-\iota)^{\frac{1}{2}A_2-1}\{v_2+z_2(2v_2)^{\frac{1}{2}}+\tfrac{1}{2}z_2^2+\tfrac{1}{2}u_2^2\}^{\frac{1}{2}A_2-1}\{(2v_2)^{\frac{1}{2}}+z_2\}. \quad (8.8) \end{aligned}$$

It will be found that the relation $\partial L/\partial u_2 = 0$ reduces in virtue of (8.7) to

$$\begin{aligned} &u_2\{-(\sigma'-\nu')+c(2v_2)^{\frac{1}{2}}z_2+\tfrac{1}{2}c(z_2^2+u_2^2)+\iota bv\} \\ &\quad = \pm\iota(fg)^{\frac{1}{2}}A_2v^{\frac{1}{2}A_1}(-\iota)^{\frac{1}{2}A_2-1}u_2\{v_2+z_2(2v_2)^{\frac{1}{2}}+\tfrac{1}{2}z_2^2+\tfrac{1}{2}u_2^2\}^{\frac{1}{2}A_2-1}, \end{aligned}$$

and is thus a consequence of (8.6).

The constants c_6, c_7, $ac_3'+c_2c_4$, $c_4^2+ac_{10}$ which have been assumed in § 7 to be in general non-zero are here equal to

$$(-\iota)^{A_2}v_2^{\frac{1}{2}A_2}f, \quad v_2^{\frac{1}{2}A_2}g, \quad -2v_2(cA_2/A_1+b), \quad 2v_2(ac-b^2)$$

respectively. Thus for general values of μ_1, μ_2, a, b, c, f, g, the solution of the conditions (8.6) may be carried out as in § 7, and we obtain two families of periodic solutions branching from Family I at the generating solution.

§ 9. *Periodic Solutions when* $\lambda = 0$.

The contact transformation (5.20), when applied to the system (5.1), yields the system (5.21):

$$\frac{dz_k}{dt}=\frac{\partial K}{\partial u_k}, \quad \frac{du_k}{dt}=-\frac{\partial K}{\partial z_k} \quad (k=1,2); \quad \ldots \ldots \quad (9.1)$$

here K is now a power series in the z_k, u_k, with coefficients linear in $(\sigma-\nu)$, and its leading terms are of the form (5.29), with at least one of c_1, d_1, c_2, d_2 non-zero:

$$K \equiv (\sigma-\nu)(c_1z_1+d_1u_1+c_2z_2+d_2u_2)+az_2u_1+\tfrac{1}{2}bz_1^2+cz_1z_2+\tfrac{1}{2}dz_2^2+\ldots. \quad (9.2)$$

Since K does not involve t it is an integral. The equations (9.1) possess also the integral F, since they are a transformation of (5.1), and hence also the integral M, $=$ K $-$F. Now M is the part of K factored by σ, viz.—

$$M \equiv \sigma(c_1z_1+d_1u_1+c_2z_2+d_2u_2+\ldots).$$

Substituting in the identity

$$\frac{\partial K}{\partial z_1}\frac{\partial M}{\partial u_1}-\frac{\partial K}{\partial u_1}\frac{\partial M}{\partial z_1}+\frac{\partial K}{\partial z_2}\frac{\partial M}{\partial u_2}-\frac{\partial K}{\partial u_2}\frac{\partial M}{\partial z_2}=0 \quad \ldots \ldots \quad (9.3)$$

the series for K, M, the vanishing of the coefficients of z_1, z_2, u_1 gives the relations

$$\left.\begin{aligned} bd_1 + cd_2 \qquad &= 0 \\ cd_1 + dd_2 - ac_1 &= 0 \\ ad_2 \qquad &= 0 \end{aligned}\right\}, \qquad (9.4)$$

whence either $c_1 = d_1 = d_2 = 0$ or $a^2b = 0$.

If $a^2b \neq 0$, this being the general case, we must thus have $c_2 \neq 0$.

If $a = 0$, $bd - c^2 \neq 0$ we have $d_1 = d_2 = 0$, and since then one of c_1, c_2 is non-zero it is (since the quadratic terms in (9.2) are symmetrical with respect to z_1, z_2) only a matter of notation to suppose $c_2 \neq 0$.

If $b = 0$ and either $a \neq 0$ or $c \neq 0$ we have from (9.4) $d_2 = 0$. Thus if $c_2 = 0$, either $c_1 \neq 0$ or $d_1 \neq 0$. If $c_1 \neq 0$ the contact transformation

$$z_1 = z'_1 + z'_2, \quad z_2 = z'_2, \quad u_1 = u'_1, \quad u_2 = -u'_1 + u'_2 \qquad (9.5)$$

gives

$$K = (\sigma - \nu)(c_1 z'_1 + c_1 z'_2 + d_1 u'_1) + az'_2 u'_1 + cz'_1 z'_2 + (c + \tfrac{1}{2}d)z'^2_2 + \ldots,$$

which is of the same form as (9.2) with $c_2 \neq 0$. If $d_1 \neq 0$ we similarly transform K to a form in which $c_2 \neq 0$ by the contact transformation

$$z_1 = z_1', \; u_1 = u_1' + z_2', \qquad z_2 = z_2', \; u_2 = u_2' + z_1'. \qquad (9.6)$$

If $b = a = c = 0$, then (9.4) gives $d = 0$ or $d_2 = 0$. In the former case one of c_1, d_1, c_2, d_2 is non-zero, and it is only a matter of notation to suppose that $c_2 \neq 0$; if in the latter case $c_2 = 0$, one of the contact transformations (9.5), (9.6) will bring K to a form similar to (9.2) in which $c_2 \neq 0$.

Thus we may always suppose that in (9.2) $c_2 \neq 0$. The identity (9.3) then shows that in the neighbourhood of the origin the equation $\partial K/\partial u_2 = 0$ is a consequence of

$$\frac{\partial K}{\partial z_1} = \frac{\partial K}{\partial u_1} = \frac{\partial K}{\partial z_2} = 0. \qquad (9.7)$$

To an equilibrium solution for the z_k, u_k corresponds by (5.20) a periodic solution for the x_k, y_k of parameter σ, and from what has just been seen the equilibrium solutions of (9.1) are those sets of constant values of the z_k, u_k satisfying (9.7). Suppose these equations solved for z_1, u_1, z_2 in terms of $(\sigma - \nu)$, u_2; then the resulting periodic solution for the x_k, y_k depends on the arbitrary parameters σ, u_2, and, as in § 7, we may show that an alteration in u_2 is equivalent to the addition of a constant to t, and so that there is no loss of generality in finding only those solutions of (9.2) for which $u_2 = 0$. Inserting the series (9.2) for K the conditions (9.7) are then

$$\left.\begin{aligned} c_1(\sigma - \nu) + bz_1 + cz_2 + \ldots \qquad &= 0 \\ d_1(\sigma - \nu) + az_2 + \ldots \qquad &= 0 \\ c_2(\sigma - \nu) + au_1 + cz_1 + dz_2 + \ldots &= 0 \end{aligned}\right\} \qquad (9.8)$$

If $a^2b \neq 0$ these can be solved for z_1, u_1, z_2 as power series in $(\sigma - \nu)$, and, substituting in (5.20), we have a singly-infinite family of periodic solutions for the x_k, y_k, which appear as Laurent series in $e^{\sigma t}$ with coefficients power series in $(\sigma - \nu)$; the generating solution is the member of this family for which $\sigma - \nu = z_1 = u_1 = z_2 = 0$.

If $acc_1 + (bd - c^2)\, d_1 - abc_2$ is non-zero we may solve for $(\sigma - \nu)$, z_1, z_2 as power series in u_1, and there is again one family of periodic solutions.

In all other cases the solution is multiple, and there is more than one family. For instance, if $a = 0$, $bd - c^2 \neq 0$ (so that $d_1 = 0$), the first and third of (9.8) are soluble for z_1, z_2 as power series in u_1, $(\sigma - \nu)$, and the second becomes a power series in u_1, $(\sigma - \nu)$, in which the terms of lowest degree are quadratic; in general this gives two distinct solutions for u_1 as power series in $(\sigma - \nu)$, and so we obtain two families of periodic solutions having the generating solution in common. For each family the x_k, y_k are Laurent series in $e^{\sigma t}$ with coefficients power series in $(\sigma - \nu)$.

A case which is specially to be noted is that in which the equations (9.8) are not independent, so that, say, when the first and third are solved for z_1, z_2 as power series in u_1, $(\sigma - \nu)$, substitution in the second makes it an identity. We then obtain a doubly-infinite family of periodic solutions for the x_k, y_k, depending on the two parameters σ, u_1. This case occurs when the equations (5.1) are a "soluble" system of which the generating solution is an "ordinary" periodic solution.

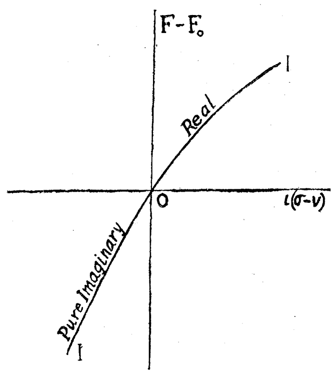
The Exponents of the Periodic Solutions.—We have seen in § 5 that the exponents of a periodic solution for the x_k, y_k are the same as those of the corresponding solution for the z_k, u_k, which is here an equilibrium solution. These are given by the investigation of § 7 (ii) if we write M, K, z_k, u_k in place of N, L, Z_k, U_k respectively, so the non-zero exponents of any periodic solution as just investigated are $\pm\,\mu$ where

$$\left(\frac{\partial M}{\partial z_2}\right)^2 \mu^2 = \begin{vmatrix} \dfrac{\partial^2 K}{\partial z_1^2}, & \dfrac{\partial^2 K}{\partial z_1\,\partial u_1}, & \dfrac{\partial^2 K}{\partial z_1\,\partial z_2}, & \dfrac{\partial M}{\partial z_1} \\ \dfrac{\partial^2 K}{\partial z_1\,\partial u_1}, & \dfrac{\partial^2 K}{\partial u_1^2}, & \dfrac{\partial^2 K}{\partial u_1\,\partial z_2}, & \dfrac{\partial M}{\partial u_1} \\ \dfrac{\partial^2 K}{\partial z_1\,\partial z_2}, & \dfrac{\partial^2 K}{\partial u_1\,\partial z_2}, & \dfrac{\partial^2 K}{\partial z_2^2}, & \dfrac{\partial M}{\partial z_2} \\ \dfrac{\partial M}{\partial z_1}, & \dfrac{\partial M}{\partial u_1}, & \dfrac{\partial M}{\partial z_2}, & 0 \end{vmatrix}, \qquad (9.9)$$

and z_1, u_1, z_2, u_2 are to be given their constant values appropriate to the solution in question. We thus obtain μ^2 as a power series in integral or fractional powers of $(\sigma - \nu)$ according as the solution of (9.8) is in integral or fractional powers of $(\sigma - \nu)$. The absolute term in this series is easily shown to vanish in virtue of (9.4), as, of course, it must since for the generating solution $\mu^2 = 0$.

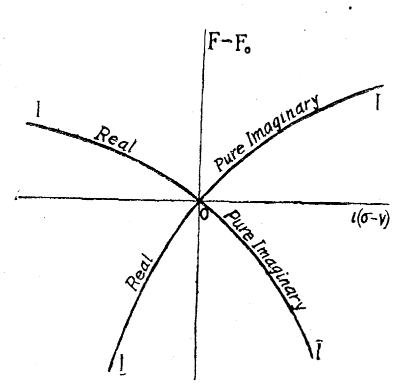
Reality of the Solutions.—We have seen (§ 5, p. 157) that when the generating solution is real and of real period and $\lambda = 0$, the transformation (5.20) may always be supposed a real transformation. Thus in the conditions (9.8) and in the expression (9.9) for μ^2 the series which occur are power series in z_1, u_1, z_2, $\iota(\sigma - \nu)$ with real coefficients. Thus the solution of (9.8) when it is unique gives z_1, u_1, z_2 as real functions of $\iota(\sigma - \nu)$, and when it is multiple unreal solutions occur in conjugate pairs. Moreover, for a real periodic solution μ^2 is real, so the non-zero exponents are either real or pure-imaginary. In general the leading term in μ^2 is linear in $(\sigma - \nu)$, so μ^2 changes sign with $\iota(\sigma - \nu)$; thus, as we pass along the (real) family of periodic solutions, the exponents change from real to pure imaginary at the generating solution.

The following figures, in which each curve corresponds to a real family of periodic solutions, are analogous to those at the end of § 7 :—



$\lambda = 0, \; a^2 b \neq 0$

Fig. 6.



$\lambda = 0, \; a = 0, \; bd - c^2 \neq 0,$

Fig. 7.

Fig. 7 illustrates the case in which there are two real solutions for z_1, u_1, z_2 in powers of $\iota(\sigma - \nu)$.

§ 10. *Direct Construction of Periodic Solutions in the Neighbourhood of a Known Periodic Solution. Convergence of the Series.*

In default of a proof that the formal power series (5.13) giving the general solution of the equations (5.1) are convergent, the investigation of §§ 5–9 is of only formal validity. In point of fact the series (5.13) are in general divergent, but the arguments by which this may be proved are of an indirect nature.* A new line of attack is therefore needed to prove the convergence of the series giving the formally periodic solutions. The procedure is to give a process for the direct construction of these solutions, and then to prove that the series involved are convergent by the method of "dominant series." The *process* makes no direct appeal to the Hamiltonian form of the equations, but to show that it is successful in constructing periodic solutions we invoke the existence theory of §§ 5–9, and to this the Hamiltonian form of the equations is, of course, essential.

* The argument is briefly that any system for which the transformation (5.20) is convergent is "soluble," so that its periodic solutions are in general "ordinary," whereas in general a Hamiltonian system possesses no ordinary periodic solutions.

Hypotheses.—We suppose (i) that for the equations (5.1), viz.,

$$\frac{dx_k}{dt} = \frac{\partial F}{\partial y_k}, \quad \frac{dy_k}{dt} = -\frac{\partial F}{\partial x_k} \qquad (k = 1, 2), \quad \dots \dots \quad (10.1)$$

we know a periodic solution of parameter ν, and exponents ± 0, $\pm \lambda$, and also (ii) that we know the normalising linear contact transformation appropriate to this solution, say

$$x_k = \theta_k(\zeta_r, \omega_r, \gamma e^{\nu t}), \quad y_k = \chi_k(\zeta_r, \omega_r, \gamma e^{\nu t}) ; \quad \dots \dots \quad (10.2)$$

under this transformation the equations (10.1) become of the "normal" form

$$\frac{d\zeta_k}{dt} = \frac{\partial G}{\partial \omega_k}, \quad \frac{d\omega_k}{dt} = -\frac{\partial G}{\partial \zeta_k}, \quad \dots \dots \quad (10.3)$$

in which G is a power series in the ζ_k, ω_k with periodic coefficients, the terms of lowest degree being quadratic and of the form (5.7), (5.9) or (5.12) in the various possible cases. We suppose, further, (iii) that the Laurent series in $\gamma e^{\nu t}$ which occur in (10.2)* are absolutely and uniformly convergent† for a certain positive range of $|\gamma e^{\nu t}|$:

$$\delta_1 \leq |\gamma e^{\nu t}| \leq \delta_2, \quad \dots \dots \quad (10.4)$$

where, when ν is pure-imaginary, we shall suppose $\delta_1 < 1 < \delta_2$. We suppose, finally, (iv) that on substituting from (10.2) in F (x_k, y_k), we obtain a series in positive powers of the ζ_k, ω_k and positive and negative powers of $\gamma e^{\nu t}$, absolutely and uniformly convergent† when

$$|\zeta_k| \leq \zeta_k^0 > 0, \qquad |\omega_k| \leq \omega_k^0 > 0, \qquad \delta_1 \leq |\gamma e^{\nu t}| \leq \delta_2 ;$$

when this is so we shall say that F *is regular in the neighbourhood of the generating solution* (which is specified by putting $\zeta_k = \omega_k = 0$ in (10.2)).

With these hypotheses we show (i) how to construct formal Laurent series giving the periodic solutions of the various Families; (ii) how to construct the normalising transformation appropriate to any such solution; (iii) that the series here involved have a region of convergence of the same nature as that supposed for the corresponding series (belonging to the generating solution) from which we started; and (iv) that F is regular in the neighbourhood of any such solution.

We call a periodic solution for which the convergency conditions (iii), (iv) are

* If, *e.g.*, $x_1 = A\zeta_1 + B\omega_1 + C\zeta_2 + D\omega_2 + E$, the Laurent series in question are the coefficients A, B, C, D, E.

† A Laurent series in $\gamma e^{\nu t}$, absolutely convergent when $\delta_1 \leq |\gamma e^{\nu t}| \leq \delta_2$, is the sum of a power series in $\gamma e^{\nu t}$, absolutely convergent when $|\gamma e^{\nu t}| \leq \delta_2$, and a power series in $(\gamma e \;)^{-1}$, absolutely convergent when $|\gamma e^{\nu t}| \geq \delta_1$. Hence, from the absolute convergence of these series follows their uniform convergence.

satisfied *regular*, and we shall show then that when the generating solution is regular the families furnished by the formal process of §§ 5–9 consist of *regular* periodic solutions.

Preliminary Transformation.—We apply to (10.1) in place of (10.2) the linear transformation

$$x_k = \theta_k(\zeta_r, \omega_r, \gamma e^{\sigma t}), \qquad y_k = \chi_k(\zeta_r, \omega_r, \gamma e^{\sigma t}), \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (10.5)$$

where σ is an arbitrary parameter. This is a contact transformation, since (10.2) is a contact transformation, and since in the conditions $[\zeta_1, \omega_1] = 1$, etc., the quantity $\gamma e^{\nu t}$ appears as an arbitrary parameter.

The resulting equations are of the form

$$\frac{d\zeta_k}{dt} = \frac{\partial H}{\partial \omega_k} + (\sigma - \nu)\frac{\partial N}{\partial \omega_k}, \qquad \frac{d\omega_k}{dt} = -\frac{\partial H}{\partial \zeta_k} - (\sigma - \nu)\frac{\partial N}{\partial \zeta_k}, \quad . \quad . \quad . \quad (10.6)$$

in which $H = F + \nu N$, and N is defined by the consistent equations

$$\sigma\frac{\partial N}{\partial \zeta_k} = [\zeta_k, t]_{xy}, \qquad \sigma\frac{\partial N}{\partial \omega_k} = [\omega_k, t]_{xy}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (10.7)$$

the functions with respect to which the Lagrange brackets are formed being (10.5); H is a power series and N is a quadratic function (not homogeneous) in the ζ_k, ω_k with coefficients which are Laurent series in $\gamma e^{\sigma t}$, and the equations involve σ only through this argument and through the factor $(\sigma - \nu)$ written explicitly in (10.6).

Now from (10.5) we obtain the transformation (5.20) if we replace the ζ_k, ω_k by certain power series in the z_k, u_k (with coefficients which are Laurent series in $\gamma e^{\sigma t}$), and the leading terms in these series are

$$\zeta_k = z_k, \quad \omega_k = u_k.$$

It follows that as far as the terms quadratic in $(\sigma - \nu)$ and the ζ_k, ω_k the Hamiltonian function $H + (\sigma - \nu) N$ is identical with the K of equations (5.21), with the trivial difference that ζ_k, ω_k are written for z_k, u_k. For when $\sigma = \nu$ (10.5) is identical with (10.2), so H is identical with G, and the quadratic terms of H are given by (5.7), (5.9) or (5.12); and for any value of σ the absolute terms in $[\zeta_k, t]$, $[\omega_k, t]$ depend only on the terms in (10.5) of degree less than 2 in the ζ_k, ω_k, and hence from (10.7) the terms of lowest degree in $(\sigma - \nu)$ N, viz., the terms linear in the ζ_k, ω_k, are the same as the corresponding terms in K. Taking for definiteness the case $\lambda \neq 0$, $\lambda \neq \frac{1}{2}\nu$, we have then as far as the leading terms (with the notation of § 5)—

$$H = \lambda\zeta_1\omega_1 - \tfrac{1}{2}a\zeta_2^2 + \ldots$$

$$N = c_2\zeta_2 + d_2\omega_2 + \ldots.$$

Let $\rho = \lambda\sigma/\nu$; then $\lambda = \rho - \lambda(\sigma - \nu)/\nu$, and we have as far as the terms quadratic in $(\sigma - \nu)$ and the ζ_k, ω_k

$$H = \rho\zeta_1\omega_1 - \tfrac{1}{2}a\zeta_2^2 + \ldots .$$

The equations (10.6) are thus explicitly

$$\left.\begin{aligned} &\frac{d\zeta_1}{dt} - \rho\zeta_1 = \phi_1(\zeta_k, \omega_k, \gamma e^{\sigma t}, \sigma - \nu), && \frac{d\omega_1}{dt} + \rho\omega_1 = \psi_1(\zeta_k, \omega_k, \gamma e^{\sigma t}, \sigma - \nu) \\ &\frac{d\zeta_2}{dt} = d_2(\sigma - \nu) + \phi_2(\zeta_k, \omega_k, \gamma e^{\sigma t}, \sigma - \nu), && \frac{d\omega_2}{dt} = a\zeta_2 - c_2(\sigma - \nu) + \psi_2(\zeta_k, \omega_k, \gamma e^{\sigma t}, \sigma - \nu) \end{aligned}\right\}, \quad \ldots \quad (10.8)$$

where the ϕ_k, ψ_k are power series in $(\sigma - \nu)$ and the ζ_k, ω_k with coefficients which are Laurent series in $\gamma e^{\sigma t}$; their terms of lowest degree are at least quadratic in $(\sigma - \nu)$ and the ζ_k, ω_k, and no power of $(\sigma - \nu)$ above the first occurs. Here if $a \neq 0$ we know from §§ 6, 7 that $c_2 \neq 0$, $d_2 = 0$, while if $a = 0$ we may always suppose that $c_2 \neq 0$.

Periodic Solutions of Family I.—We desire a solution of (10.8) in which the ζ_k, ω_k are Laurent series in $\gamma e^{\sigma t}$. For this purpose substitute

$$\left.\begin{aligned} \zeta_k &= \zeta_k^{(1)} + \zeta_k^{(2)} + \ldots \\ \omega_k &= \omega_k^{(1)} + \omega_k^{(2)} + \ldots \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots \quad (10.9)$$

in the equations (10.8) and equate terms of the same order, regarding $(\sigma - \nu)$ as of order 1 and the $\zeta_k^{(r)}$, $\omega_k^{(r)}$ as of order r. This gives

$$\left.\begin{aligned} &\frac{d\zeta_1^{(1)}}{dt} = \rho\zeta_1^{(1)}, && \frac{d\omega_1^{(1)}}{dt} = -\rho\omega_1^{(1)} \\ &\frac{d\zeta_2^{(1)}}{dt} = d_2(\sigma - \nu), && \frac{d\omega_2^{(1)}}{dt} = a\zeta_2^{(1)} - c_2(\sigma - \nu) \end{aligned}\right\}, \quad \ldots\ldots \quad (10.10)$$

$$\left.\begin{aligned} \frac{d\zeta_1^{(r)}}{dt} - \rho\zeta_1^{(r)} &= f_1^{(r)}(\zeta_k^{(1)}, \omega_k^{(1)}, \ldots \zeta_k^{(r-1)}, \omega_k^{(r-1)}, \gamma e^{\sigma t}, \sigma - \nu) \\ \frac{d\omega_1^{(r)}}{dt} + \rho\omega_1^{(r)} &= g_1^{(r)}(\zeta_k^{(1)}, \omega_k^{(1)}, \ldots \zeta_k^{(r-1)}, \omega_k^{(r-1)}, \gamma e^{\sigma t}, \sigma - \nu) \\ \frac{d\zeta_2^{(r)}}{dt} &= f_2^{(r)}(\zeta_k^{(1)}, \omega_k^{(1)}, \ldots \zeta_k^{(r-1)}, \omega_k^{(r-1)}, \gamma e^{\sigma t}, \sigma - \nu) \\ \frac{d\omega_2^{(r)}}{dt} &= a\zeta_2^{(r)} + g_2^{(r)}(\zeta_k^{(1)}, \omega_k^{(1)}, \ldots \zeta_k^{(r-1)}, \omega_k^{(r-1)}, \gamma e^{\sigma t}, \sigma - \nu) \end{aligned}\right\}, \quad \ldots \quad (10.11)$$

where the $f_k^{(r)}$, $g_k^{(r)}$ are rth order polynomials whose coefficients are Laurent series in $\gamma e^{\sigma t}$.

First Method of Solution.—The general solution of (10.10) is

$$\zeta_1^{(1)} = \alpha_1 e^{\rho t}, \qquad \omega_1^{(1)} = \beta_1 e^{-\rho t}, \qquad \zeta_2^{(1)} = \alpha_2 + d_2(\sigma - \nu)\, t,$$
$$\omega_2^{(1)} = \beta_2 - c_2(\sigma - \nu)\, t + a\alpha_2 t + \tfrac{1}{2} a d_2 (\sigma - \nu)\, t^2,$$

where α_1, β_1, α_2, β_2 are arbitrary constants, but they cannot be left all arbitrary if the solution is to have the desired form in which t enters only through the argument $\gamma e^{\sigma t}$; we must, in fact, have

$$\alpha_1 = \beta_1 = 0, \qquad d_2(\sigma - \nu) = 0, \qquad a\alpha_2 - c_2(\sigma - \nu) = 0,$$

and here we know that the third condition is in general satisfied of itself, while the others give α_1, β_1, α_2 in terms of $(\sigma - \nu)$. The right-hand sides of the equations for the $\zeta_k^{(2)}$, $\omega_k^{(2)}$ are now known functions of t, and we may solve for the $\zeta_k^{(2)}$, $\omega_k^{(2)}$, and so on.

Suppose that the $\zeta_k^{(1)}$, $\omega_k^{(1)}$, ... $\zeta_k^{(r-1)}$, $\omega_k^{(r-1)}$ have been obtained as Laurent series in $\gamma e^{\sigma t}$; then the right-hand sides of (10.11) become known Laurent series in $\gamma e^{\sigma t}$, and a solution for $\zeta_k^{(r)}$, $\omega_k^{(r)}$ is obtained by adding the "particular integral" solutions corresponding to each term on the right.* To a term $Ae^{n\sigma t}$ on the right of the equation for $\zeta_1^{(r)}$ corresponds the particular integral

$$\zeta_1^{(r)} = Ae^{n\sigma t}/(n\sigma - \rho);$$

to the same term in the equation for $\omega_1^{(r)}$ corresponds

$$\omega_1^{(r)} = Ae^{n\sigma t}/(n\sigma + \rho);$$

and to the same term in the equation for $\zeta_2^{(r)}$ or $\omega_2^{(r)}$ corresponds

$$\zeta_2^{(r)} \text{ or } \omega_2^{(r)} = Ae^{n\sigma t}/n\sigma;$$

these forms are valid, provided $n\sigma \neq \rho$, $n\sigma \neq -\rho$, $n\sigma \neq 0$ respectively. If these conditions are not satisfied the particular integral is of the form $Ate^{n\sigma t}$; for instance, the solution of $d\zeta_2^{(r)}/dt = A$ is $\zeta_2^{(r)} = At$. Such terms, in which t enters other than through exponential functions, are called *secular terms*, and a term on the right of an equation which on integration leads to a secular term is called a *critical term*. Since our object is to construct a solution free of secular terms we must, by some means, ensure that no critical terms occur on the right at the successive approximations.

This condition is satisfied of itself as regards the equations for $\zeta_1^{(r)}$, $\omega_1^{(r)}$, for here a term $Ae^{n\sigma t}$ is critical only if $n\sigma = \pm\rho$, and this cannot be so since $\rho/\sigma = \lambda/\nu$ and λ is not an integral multiple of ν.† The usual method of securing its satisfaction as regards the equations for $\zeta_2^{(r)}$, $\omega_2^{(r)}$ is in principle as follows: we include in the solution for $\zeta_2^{(2)}$, $\omega_2^{(2)}$, $\zeta_2^{(3)}$, $\omega_2^{(3)}$, ... the "complementary function" terms which are, in fact, additive arbitrary constants; in any equation the coefficients of the critical terms depend

* Of course, we cannot solve for $\omega_2^{(r)}$ until $\zeta_2^{(r)}$ has been found.

† By definition we have $0 \leq R(\lambda/\nu) \leq \frac{1}{2}$, and we are here treating the case in which $\lambda \neq 0$.

upon arbitrary constants which have been thus introduced in solving *preceding* equations, and their vanishing is secured by suitably assigning these arbitraries.* Thus we have already seen that to make the critical term vanish from the equation for $\omega_2^{(1)}$ we must put $\alpha_2 = c_2(\sigma - \nu)/a$, α_2 being the "complementary function" term in $\zeta_2^{(1)}$.

The disadvantage of this procedure is that the expressions for the $\zeta_k^{(r)}$, $\omega_k^{(r)}$ are not completely and finally determined each from a single equation, and the presence of constants whose values are fixed only when we are considering subsequent equations introduces grave complications when we attempt to prove the series convergent. Our problem therefore is to determine the $\zeta_k^{(r)}$, $\omega_k^{(r)}$ each *completely* from the appropriate equation of (10.11), and at the same time to obtain for them expressions free of secular terms. It is solved as follows:

Second Method of Solution.—We do not attempt by the adjustment of previously introduced arbitrary constants to make the coefficients of the critical terms on the right of (10.11) vanish. Instead of this we simply *omit* the critical terms and obtain the $\zeta_k^{(r)}$, $\omega_k^{(r)}$ by adding the "particular integral" solutions corresponding to each term which remains, no arbitrary "complementary function" terms being added except at the first stage, *i.e.*, when solving (10.10). The expressions thus obtained for the $\zeta_k^{(1)}$, $\omega_k^{(1)}$, ... $\zeta_k^{(r-1)}$, $\omega_k^{(r-1)}$ are substituted in the equations for the $\zeta_k^{(r)}$, $\omega_k^{(r)}$, which are then treated similarly. Of course the series (10.9) as thus term-by-term constructed do not satisfy the equations (10.8), but from their mode of construction they will evidently, when substituted therein, make the right- and left-hand sides identical *except for series of critical terms*, which appear on the right unbalanced by anything on the left. Equating to zero these series of critical terms, we obtain a set of conditions (actually two conditions of which one is a consequence of the other), and (10.9) will be a solution provided the constants, hitherto arbitrary, on which these series depend, satisfy these conditions. As in the first method, it is necessary that the constants of integration should not all remain arbitrary, but the necessary restriction is imposed on them as a final step and not during the construction of the solution.

To proceed to details: For the first two of (10.10) the solution which has the desired form in which t enters only through positive or negative integral powers of $\gamma e^{\sigma t}$ is $\zeta_1^{(1)} = \omega_1^{(1)} = 0$, which arises from the general solution $\zeta_1^{(1)} = \alpha_1 e^{\rho t}$, $\omega_1^{(1)} = \beta_1 e^{-\rho t}$ by putting $\alpha_1 = \beta_1 = 0$. In the third equation we reject the critical term $d_2(\sigma - \nu)$ and obtain $\zeta_2^{(1)} = \alpha_2$, and in the fourth, which is now

$$\frac{d\omega_2^{(1)}}{dt} = a\alpha_2 - c_2(\sigma - \nu),$$

we reject the critical terms $a\alpha_2 - c_2(\sigma - \nu)$ and obtain $\omega_2^{(1)} = \beta_2$; here α_2, β_2 are constants at present arbitrary, and σ is, of course, the arbitrary parameter which made its appearance in (10.5).

Suppose we have obtained the $\zeta_k^{(2)}$, $\omega_k^{(2)}$, ... $\zeta_k^{(r-1)}$, $\omega_k^{(r-1)}$ as homogeneous polynomials

* For numerous examples, see "Méth. Nouv.," and MOULTON, 'Periodic Orbits.'

in α_2, β_2, $(\sigma - \nu)$ of degrees 2, 2, ... $(r-1)$, $(r-1)$ respectively, the coefficients being Laurent series in $\gamma e^{\sigma t}$. Then the equations (10.11) become of the form

$$\left.\begin{aligned} \frac{d\zeta_1^{(r)}}{dt} - \rho\zeta_1^{(r)} &= p_1^{(r)}(\alpha_2, \beta_2, \gamma e^{\sigma t}, \sigma - \nu) \\ \frac{d\omega_1^{(r)}}{dt} + \rho\omega_1^{(r)} &= q_1^{(r)}(\alpha_2, \beta_2, \gamma e^{\sigma t}, \sigma - \nu) \\ \frac{d\zeta_2^{(r)}}{dt} &= p_2^{(r)}(\alpha_2, \beta_2, \gamma e^{\sigma t}, \sigma - \nu) + u_2^{(r)}(\alpha_2, \beta_2, \gamma e^{\sigma t}, \sigma - \nu) \\ \frac{d\omega_2^{(r)}}{dt} &= a\zeta_2^{(r)} + q_2^{(r)}(\alpha_2, \beta_2, \gamma e^{\sigma t}, \sigma - \nu) + v_2^{(r)}(\alpha_2, \beta_2, \gamma e^{\sigma t}, \sigma - \nu) \end{aligned}\right\}, \quad (10.12)$$

where the $p_k^{(r)}$, $q_k^{(r)}$ are aggregates of terms none of which is critical, and $u_2^{(r)}$, $v_2^{(r)}$ are aggregates of critical terms; the $p_k^{(r)}$, $q_k^{(r)}$ are homogeneous polynomials of degree r in α_2, β_2, $(\sigma - \nu)$ whose coefficients are Laurent series in $\gamma e^{\sigma t}$, and $u_2^{(r)}$, $v_2^{(r)}$ are homogeneous polynomials in α_2, β_2, $(\sigma - \nu)$.* As we have seen, the first two equations can contain no critical terms. We reject $u_2^{(r)}$, $v_2^{(r)}$, and if

$$p_k^{(r)} = \Sigma A_k \alpha_2{}^a \beta_2{}^b (\sigma - \nu)^c (\gamma e^{\sigma t})^n, \qquad q_k^{(r)} = \Sigma B_k \alpha_2{}^a \beta_2{}^b (\sigma - \nu)^c (\gamma e^{\sigma t})^n$$

we obtain the solution

$$\left.\begin{aligned} \zeta_1^{(r)} &= \Sigma \frac{A_1 \alpha_2{}^a \beta_2{}^b (\sigma - \nu)^c (\gamma e^{\sigma t})^n}{n\sigma - \rho}, \qquad \omega_1^{(r)} = \Sigma \frac{B_1 \alpha_2{}^a \beta_2{}^b (\sigma - \nu)^c (\gamma e^{\sigma t})^n}{n\sigma + \rho} \\ \zeta_2^{(r)} &= \Sigma \frac{A_2 \alpha_2{}^a \beta_2{}^b (\sigma - \nu)^c (\gamma e^{\sigma t})^n}{n\sigma} \\ \omega_2^{(r)} &= \Sigma \left(\frac{aA_2}{n^2\sigma^2} + \frac{B_2}{n\sigma}\right) \alpha_2{}^a \beta_2{}^b (\sigma - \nu)^c (\gamma e^{\sigma t})^n \end{aligned}\right\}. \quad (10.13)$$

No "complementary function" terms with arbitrary coefficients are introduced.† We proceed similarly to determine the $\zeta_k^{(r+1)}$, $\omega_k^{(r+1)}$, and so on.

When the series (10.9), as thus constructed, are substituted in the equations (10.8) they make the right- and left-hand sides identical, except for series of critical terms, which appear on the right unbalanced by anything on the left. In the first two equations there appear no critical terms, so these are satisfied; in the third and fourth the unbalanced critical terms are power series in α_2, β_2, $(\sigma - \nu)$, viz., from (10.12)

$$\left.\begin{aligned} u_2^{(1)} + u_2^{(2)} + u_2^{(3)} + \dots \\ v_2^{(1)} + v_2^{(2)} + v_2^{(3)} + \dots \end{aligned}\right\}, \quad \dots\dots\dots\dots \quad (10.14)$$

* We write $u_2^{(r)}$, $v_2^{(r)}$ in (10.12) as depending on the argument $\gamma e^{\sigma t}$ because the polynomials with which we shall later be comparing them depend on the corresponding argument Γ (see below). Actually, of course, $u_2^{(r)}$, $v_2^{(r)}$ consist of just those terms on the right of the equations in which $\gamma e^{\sigma t}$ is raised to the power *zero*.

† To add to $\zeta_1^{(r)}$ the term $\alpha_1 e^{\rho t}$ would spoil the desired form of the solution, while to add to $\zeta_2^{(r)}$ or $\omega_2^{(r)}$ an arbitrary integration-constant would amount in the completed series merely to an alteration in the already arbitrary α_2 or β_2.

where, as we have seen,

$$u_2^{(1)} = d_2(\sigma - \nu), \qquad v_2^{(1)} = a\alpha_2 - c_2(\sigma - \nu).$$

We have therefore in (10.9) a solution of the equations (10.8), provided

$$\sum_{r=1}^{\infty} u_2^{(r)} = \sum_{r=1}^{\infty} v_2^{(r)} = 0, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (10.15)$$

to which conditions we add the conditions already introduced

$$\alpha_1 = \beta_1 = 0; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (10.16)$$

this solution is periodic and of parameter σ. On substitution of (10.9) in (10.5) we obtain for the x_k, y_k a periodic solution of parameter σ. Now from §§ 6, 7 we know that the equations (10.1) possess formal periodic solutions precisely of the nature we have been constructing, and we infer that these solutions must be obtainable from (10.9) by restricting the α_k, β_k, $(\sigma - \nu)$ to satisfy (10.15), (10.16). In fact the conditions (10.15), (10.16) render (10.9) a periodic solution, just as in §§ 6, 7 the conditions

$$\frac{\partial K}{\partial u_2} = \frac{\partial K}{\partial z_2} = z_1 = u_1 = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (10.17)$$

make (5.20) a periodic solution of (5.1). We see that α_k, β_k correspond respectively to z_k, u_k, and that the conditions (10.15), (10.16) correspond to the conditions (10.17).* We infer that of the conditions (10.15) the first is a consequence of the second, and that there is no loss of generality in putting $\beta_2 = 0$ in the second. This may then be solved for α_2 in integral or fractional powers of $(\sigma - \nu)$, according as $a \neq 0$ or $a = 0$; in either case we obtain one family (Family I) of periodic solutions whose parameter σ varies continuously along the family; the generating solution appears as the member of the family for which $\sigma = \nu$.

Periodic Solutions of Families II, III.—These, as we know, exist when $\lambda/\nu = \lambda_0/\nu_0$, the ratio of two integers; when this is so we have in (10.8) $\rho/\sigma = \lambda_0/\nu_0$. In the proposed solution we now allow t to occur through positive or negative powers of either $e^{\rho t}$ or $e^{\sigma t}$, since ρ, σ are commensurable, so for the solution of (10.10) we take

$$\zeta_1^{(1)} = \alpha_1 e^{\rho t}, \qquad \omega_1^{(1)} = \beta_1 e^{-\rho t}, \qquad \zeta_2^{(1)} = \alpha_2, \qquad \omega_2^{(1)} = \beta_2,$$

where the α_k, β_k are constants at present arbitrary; we have, as before, rejected the critical terms $d_2(\sigma - \nu)$, $a\alpha_2 - c_2(\sigma - \nu)$ in the third and fourth of these equations. Suppose we have obtained the $\zeta_k^{(2)}$, $\omega_k^{(2)}, \ldots \zeta_k^{(r-1)}$, $\omega_k^{(r-1)}$ as homogeneous polynomials in the arguments $\alpha_1 e^{\rho t}$, $\beta_1 e^{-\rho t}$, α_2, β_2, $(\sigma - \nu)$ of orders $2, 2, \ldots (r-1), (r-1)$,

* It may be noted that the linear terms in these conditions have precisely the same form.

respectively, the coefficients being Laurent series in $\gamma e^{\sigma t}$; then the right-hand sides of (10.1) become similar polynomials of order r. In the equation for $\zeta_1^{(r)}$ a term

$$A(\alpha_1 e^{\rho t})^{a_1} (\beta_1 e^{-\rho t})^{b_1} \alpha_2^{a_2} \beta_2^{b_2} (\sigma - \nu)^c (\gamma e^{\sigma t})^n$$

is critical if

$$\rho(a_1 - b_1) + \sigma n = \rho; \quad \ldots \ldots \ldots \ldots \quad (10.18)$$

in the equation for $\omega_1^{(r)}$ the same term is critical if $\rho(a_1 - b_1) + \sigma n = -\rho$; and in the equations for $\zeta_2^{(r)}$, $\omega_2^{(r)}$ the same term is critical if $\rho(a_1 - b_1) + \sigma n = 0$. As before, we reject the critical terms and obtain the solution for the $\zeta_k^{(r)}$, $\omega_k^{(r)}$ by introducing in each term the appropriate denominator; in the four equations the denominators appropriate to the term (10.17) are respectively $\rho(a_1 - b_1) + \sigma n - \rho$, $\rho(a_1 - b_1) + \sigma n + \rho$, $\rho(a_1 - b_1) + \sigma n$, $\rho(a_1 - b_1) + \sigma n$.

The series (10.9) as thus constructed satisfy the equations (10.8), provided four series of unbalanced critical terms vanish. In the first equation this unbalanced series consists of terms (10.17) for each of which the indices satisfy (10.18), so the condition is of the form

$$e^{\rho t} \Sigma A \alpha_1^{a_1} \beta_1^{b_1} \alpha_2^{a_2} \beta_2^{b_2} \gamma^n (\sigma - \nu)^c = 0,$$

in which for each term the indices satisfy (10.18), *i.e.*,

$$\lambda_0 (a_1 - b_1) + \nu_0 n = \lambda_0;$$

it is thence easily shown that the condition is of the form

$$\alpha_1 P + Q/\beta_1 = 0,$$

where P, Q are power series in the arguments $\alpha_1\beta_1$, $\alpha_1^{\nu_0}\gamma^{-\lambda_0}$, $\beta_1^{\nu_0}\gamma^{\lambda_0}$, α_2, $\beta_2(\sigma - \nu)$. The other three conditions are dealt with similarly, and the set of four is seen to be precisely of the form of the conditions (7.13), together with $\partial L/\partial Z_2 = \partial L/\partial U_2 = 0$, the arguments α_2, β_2, $\alpha_1\beta_1$, $\alpha_1^{\nu_0}\gamma^{-\lambda_0}$, $\beta_1^{\nu_0}\gamma^{\lambda_0}$ corresponding respectively to Z_2, U_2, Z_1U_1, $Z_1^{\nu_0}$, $U_1^{\nu_0}$. We infer that the series (10.9), on substitution in (10.5), will furnish for the x_k, y_k families of periodic solutions precisely as required by the existence theory of § 7, provided the α_k, β_k are so determined in terms of $(\sigma - \nu)$ that the four series of critical terms vanish, and that of these four conditions we can reject that arising from the third equation and, putting $\beta_2 = 0$, solve the remainder for α_1, β_1, α_2 as power series in $(\sigma - \nu)^{\frac{1}{2}}$.

The treatment of the cases $\lambda = \frac{1}{2}\nu$, $\lambda = 0$ is entirely similar, and will not be given.

Convergence of the Series.

Notation.—Let $f(x_1, x_2, \ldots)$ be a series (finite or infinite) of positive or negative powers of arguments $x_1, x_2, \ldots$ and $F(X_1, X_2, \ldots)$ a corresponding series in corresponding arguments $X_1, X_2, \ldots$; then, if each coefficient in F is positive and not less than the modulus of the corresponding coefficient in f, we write

$$f(x_1, x_2, \ldots) \ll F(X_1, X_2, \ldots),$$

and F is said to be a *dominant* series for f.

Lemma.—Suppose that the x_k, X_k occur in f, F through positive powers only, and that by means of relations*

$$x_r = \phi_r(y_1, y_2, \ldots), \qquad X_r = \Phi_r(Y_1, Y_2, \ldots),$$

where $\phi_r(y_1, y_2, \ldots) \ll \Phi_r(Y_1, Y_2, \ldots)$ we express f, F in terms of the y_k, Y_k respectively. Then it is clear that

$$f(y_1, y_2, \ldots) \ll F(Y_1, Y_2, \ldots).$$

The Lemma remains true if only certain of the x_k, X_k are thus replaced in terms of other arguments, and there is no need then to restrict the remaining x_k, X_k to occurrence through positive powers only.

From the hypothesis relating to the result of substituting (10.2) in F it follows that on substituting (10.5) in F we obtain a series in $\gamma e^{\sigma t}$ and the ζ_k, ω_k, absolutely convergent when

$$|\zeta| \leq \zeta_k^0, \qquad |\omega_k| \leq \omega_k^0, \qquad \delta_1 \leq |\gamma e^{\sigma t}| \leq \delta_2. \quad \ldots \ldots \quad (10.19)$$

From the hypothesis relating to the convergence of the Laurent series in $\gamma e^{\nu t}$ involved in (10.2) there follows the similar convergence of the Laurent series in $\gamma e^{\sigma t}$ involved in (10.5), and hence from the definition (10.7) of N it is easily seen that the coefficients of the various powers of the ζ_k, ω_k in N are Laurent series absolutely convergent when $\delta_1 \leq |\gamma e^{\sigma t}| \leq \delta_2$. Hence the series ϕ_k, ψ_k on the right of the equations (10.8) are absolutely convergent under the conditions (10.19). For these series we find dominant series as follows:—

By separating the terms which involve positive from those which involve negative powers of $\gamma e^{\sigma t}$, and by separating the terms having $(\sigma - \nu)$ as a factor from those not having this factor, we write the ϕ_k, ψ_k as sums of four *power* series, *e.g.*,

$$\phi_1 = \Sigma\, P_{a_1b_1a_2b_2n}\zeta_1^{a_1}\omega_1^{b_1}\zeta_2^{a_2}\omega_2^{b_2}(\gamma e^{\sigma t})^n + \Sigma\, Q_{a_1b_1a_2b_2n}\zeta_1^{a_1}\omega_1^{b_1}\zeta_2^{a_2}\omega_2^{b_2}\{(\gamma e^{\sigma t})^{-1}\}^n$$
$$+ (\sigma - \nu)\,\Sigma\, R_{a_1b_1a_2b_2n}\zeta_1^{a_1}\omega_1^{b_1}\zeta_2^{a_2}\omega_2^{b_2}(\gamma e^{\sigma t})^n + (\sigma - \nu)\,\Sigma\, S_{a_1b_1a_2b_2n}\zeta_1^{a_1}\omega_1^{b_1}\zeta_2^{a_2}\omega_2^{b_2}\{(\gamma e^{\sigma t})^{-1}\}^n;$$

in these four series the terms of lowest degree in the ζ_k, ω_k are respectively of degrees 2, 2, 1, 1. The first and third of these must be absolutely convergent when

$$|\zeta_k| \leq \zeta_k^0, \qquad |\omega_k| \leq \omega_k^0, \qquad |\gamma e^{\sigma t}| \leq \delta_2,$$

so there are positive constants, T, V, such that

$$|P_{a_1b_1a_2b_2n}| \leq \frac{T}{(\zeta_1^0)^{a_1}(\omega_1^0)^{b_1}(\zeta_2^0)^{a_2}(\omega_2^0)^{b_2}\delta_2^n}, \qquad |R_{a_1b_1a_2b_2n}| \leq \frac{V}{(\zeta_1^0)^{a_1}(\omega_1^0)^{b_1}(\zeta_2^0)^{a_2}(\omega_2^0)^{b_2}\delta_2^n};$$

* The ϕ, Φ can contain negative as well as positive powers.

the second and fourth must be absolutely convergent when

$$|\zeta_k| \leq \zeta_k^0, \qquad |\omega_k| \leq \omega_k^0, \qquad |\gamma e^{\sigma t}|^{-1} \leq \delta_1^{-1},$$

so there are positive constants U, W, such that

$$|Q_{a_1 b_1 a_2 b_2 n}| \leq \frac{U}{(\zeta_1^0)^{a_1}(\omega_1^0)^{b_1}(\zeta_2^0)^{a_2}(\omega_2^0)^{b_2}\delta_1^{-n}}, \qquad |S_{a_1 b_1 a_2 b_2 n}| \leq \frac{W}{(\zeta_1^0)^{a_1}(\omega_1^0)^{b_1}(\zeta_2^0)^{a_2}(\omega_2^0)^{b_2}\delta_1^{-n}}.$$

Thus for each of the ϕ_k, ψ_k there is a dominant series of the form

$$\begin{aligned} &T\Sigma\left(\frac{\zeta_1}{\zeta_1^0}\right)^{a_1}\left(\frac{\omega_1}{\omega_1^0}\right)^{b_1}\left(\frac{\zeta_2}{\zeta_2^0}\right)^{a_2}\left(\frac{\omega_2}{\omega_2^0}\right)^{b_2}\left(\frac{\gamma e^{\sigma t}}{\delta_2}\right)^{n} + U\Sigma\left(\frac{\zeta_1}{\zeta_1^0}\right)^{a_1}\left(\frac{\omega_1}{\omega_1^0}\right)^{b_1}\left(\frac{\zeta_2}{\zeta_2^0}\right)^{a_2}\left(\frac{\omega_2}{\omega_2^0}\right)^{b_2}\left(\frac{\delta_1}{\gamma e^{\sigma t}}\right)^{n} \\ &\quad + (\sigma - \nu) V\Sigma\left(\frac{\zeta_1}{\zeta_1^0}\right)^{a_1}\left(\frac{\omega_1}{\omega_1^0}\right)^{b_1}\left(\frac{\zeta_2}{\zeta_2^0}\right)^{a_2}\left(\frac{\omega_2}{\omega_2^0}\right)^{b_2}\left(\frac{\gamma e^{\sigma t}}{\delta_2}\right)^{n} \\ &\quad\quad + (\sigma - \nu) W\Sigma\left(\frac{\zeta_1}{\zeta_1^0}\right)^{a_1}\left(\frac{\omega_1}{\omega_1^0}\right)^{b_1}\left(\frac{\zeta_2}{\zeta_2^0}\right)^{a_2}\left(\frac{\omega_2}{\omega_2^0}\right)^{b_2}\left(\frac{\delta_1}{\gamma e^{\sigma t}}\right)^{n}, \quad . \; . \; . \quad (10.20) \end{aligned}$$

and by proper choice of T, U, V, W this one series is dominant for each of ϕ_1, ψ_1, ϕ_2, ψ_2. Let δ (> 0) be the smallest of ζ_1^0, ω_1^0, ζ_2^0, ω_2^0, and let C be the greater of T, V and B the greater of U, W; then the series (10.20) is dominated by the expansion of

$$\left\{\frac{(\zeta_1+\omega_1+\zeta_2+\omega_2)^2/\delta^2 + (\sigma-\nu)(\zeta_1+\omega_1+\zeta_2+\omega_2)/\delta}{1-(\zeta_1+\omega_1+\zeta_2+\omega_2)/\delta}\right\}\left\{C\Sigma\left(\frac{\gamma e^{\sigma t}}{\delta_2}\right)^n + B\Sigma\left(\frac{\delta_1}{\gamma e^{\sigma t}}\right)^n\right\}. \quad (10.21)$$

It will be sufficient to treat in detail the convergence of the series giving the periodic solutions of Family I. Let Φ_r, Ψ_r be series in arguments Z_k, Ω_k, Γ, s (corresponding respectively to the arguments ζ_k, ω_k, $\gamma e^{\sigma t}$, $(\sigma - \nu)$) such that

$$\left.\begin{aligned} \phi_r(\zeta_k, \omega_k, \gamma e^{\sigma t}, \sigma-\nu) &\ll \Phi_r(Z_k, \Omega_k, \Gamma, s) \\ \psi_r(\zeta_k, \omega_k, \gamma e^{\sigma t}, \sigma-\nu) &\ll \Psi_r(Z_k, \Omega_k, \Gamma, s) \end{aligned}\right\} \quad (r = 1, 2); \quad . \; . \; . \quad (10.22)$$

we shall presently suppose the Φ_r, Ψ_r of the form (10.21), but at present need suppose only that they satisfy the "dominance-condition" (10.22). Then, as comparison-equations for (10.8) we take

$$\left.\begin{aligned} \varepsilon Z_1 &= \Phi_1(Z_k, \Omega_k, \Gamma, s), & \varepsilon\Omega_1 &= \Psi_1(Z_k, \Omega_k, \Gamma, s) \\ \varepsilon Z_2 &= \varepsilon A_2 + \Phi_2(Z_k, \Omega_k, \Gamma, s), & \varepsilon\Omega_2 &= \varepsilon B_2 + AZ_2 + \Psi_2(Z_k, \Omega_k, \Gamma, s) \end{aligned}\right\}, \quad (10.23)$$

where $A = |a|$ and ε is a positive constant whose value will be specified presently. These are to be solved for the Z_k, Ω_k in terms of A_2, B_2, s, Γ; the arguments A_2, B_2 correspond of course to α_2, β_2 respectively. For this purpose substitute

$$\left.\begin{aligned} Z_k &= Z_k^{(1)} + Z_k^{(2)} + \dots \\ \Omega_k &= \Omega_k^{(1)} + \Omega_k^{(2)} + \dots \end{aligned}\right\}; \quad . \; . \; . \; . \; . \; . \; . \; . \quad (10.24)$$

then, separating out terms of the same order—regarding the $Z_k^{(r)}$, $\Omega_k^{(r)}$ as of order r and A_2, B_2, s as of order 1—we find

$$\left.\begin{array}{ll} \varepsilon Z_1^{(1)} = 0, & \varepsilon\,\Omega_1^{(1)} = 0 \\ \varepsilon Z_2^{(1)} = \varepsilon A_2, & \varepsilon\,\Omega_2^{(1)} = \varepsilon B_2 + AZ_2^{(1)} \end{array}\right\}, \quad \ldots \ldots \quad (10.25)$$

$$\left.\begin{array}{l} \varepsilon Z_1^{(r)} = F_1^{(r)}(Z_k^{(1)},\ \Omega_k^{(1)}, \ldots Z_k^{(r-1)},\ \Omega_k^{(r-1)},\ \Gamma,\ s) \\ \varepsilon\Omega_1^{(r)} = G_1^{(r)}(Z_k^{(1)},\ \Omega_k^{(1)}, \ldots Z_k^{(r-1)},\ \Omega_k^{(r-1)},\ \Gamma,\ s) \\ \varepsilon Z_2^{(r)} = F_2^{(r)}(Z_k^{(1)},\ \Omega_k^{(1)}, \ldots Z_k^{(r-1)},\ \Omega_k^{(r-1)},\ \Gamma,\ s) \\ \varepsilon\Omega_2^{(r)} = AZ_2^{(r)} + G_2^{(r)}(Z_k^{(1)}, \Omega_k^{(1)}, \ldots Z_k^{(r-1)}, \Omega_k^{(r-1)},\ \Gamma,\ s) \end{array}\right\}, \quad (10.26)$$

where, from the preliminary Lemma, we have in virtue of (10.22)

$$\begin{matrix} f_k^{(r)} \\ g_k^{(r)} \end{matrix}(\zeta_k^{(1)},\ \omega_k^{(1)}, \ldots \zeta_k^{(r-1)},\ \omega_k^{(r-1)},\ \gamma e^{\sigma t},\ \sigma - \nu) \ll \begin{matrix} F_k^{(r)} \\ G_k^{(r)} \end{matrix}(Z_k^{(1)},\ \Omega_k^{(1)}, \ldots Z_k^{(r-1)},\ \Omega_k^{(r-1)},\ \Gamma,\ s). \quad (10.27)$$

The equations (10.25) give

$$Z_1^{(1)} = \Omega_1^{(1)} = 0, \qquad Z_2^{(1)} = A_2,\ \Omega_2^{(1)} = B_2 + AA_2/\varepsilon, \quad \ldots \quad (10.28)$$

and the succeeding equations are then soluble in succession for the $Z_k^{(2)}$, $\Omega_k^{(2)}$, ... as homogeneous polynomials in A_2, B_2, s of degrees 2, 2, ..., the coefficients being Laurent series in Γ. For when the $Z_k^{(2)}$, $\Omega_k^{(2)}$, ... $Z_k^{(r-1)}$, $\Omega_k^{(r-1)}$ have been so obtained, (10.26) become

$$\left.\begin{array}{l} \varepsilon Z_1^{(r)} = P_1^{(r)}(A_2,\ B_2,\ \Gamma,\ s) \\ \varepsilon\Omega_1^{(r)} = Q_1^{(r)}(A_2,\ B_2,\ \Gamma,\ s) \\ \varepsilon Z_2^{(r)} = P_2^{(r)}(A_2,\ B_2,\ \Gamma,\ s) \\ \varepsilon\Omega_2^{(r)} = AZ_2^{(r)} + Q_2^{(r)}(A_2,\ B_2,\ \Gamma,\ s) \end{array}\right\}, \quad \ldots \ldots \quad (10.29)$$

where the $P_k^{(r)}$, $Q_k^{(r)}$ are homogeneous polynomials of degree r in A_2, B_2, s with coefficients which are Laurent series in Γ, and these furnish for the $Z_k^{(r)}$, $\Omega_k^{(r)}$ expressions of the stated form. All coefficients in the series (10.24) are positive.

Suppose that we have proved that

$$\begin{matrix} \zeta_k^{(i)} \\ \omega_k^{(i)} \end{matrix}(\alpha_2,\ \beta_2,\ \gamma e^{\sigma t},\ \sigma - \nu) \ll \begin{matrix} Z_k^{(i)} \\ \Omega_k^{(i)} \end{matrix}(A_2,\ B_2,\ \Gamma,\ s) \quad \ldots \ldots \quad (10.30)$$

for $i = 1, 2, \ldots (r-1)$; then from (10.27) we have on use of the Lemma

$$\begin{matrix} p_1^{(r)} \\ q_1^{(r)} \end{matrix}(\alpha_2,\ \beta_2,\ \gamma e^{\sigma t},\ \sigma - \nu) \ll \begin{matrix} P_1^{(r)} \\ Q_1^{(r)} \end{matrix}(A_2,\ B_2,\ \Gamma,\ s), \quad \ldots \ldots \quad (10.31)$$

$$\begin{matrix} p_2^{(r)} \\ q_2^{(r)} \end{matrix}(\alpha_2, \beta_2, \gamma e^{\sigma t}, \sigma - \nu) + \begin{matrix} u_2^{(r)} \\ v_2^{(r)} \end{matrix}(\alpha_2, \beta_2, \gamma e^{\sigma t}, \sigma - \nu) \ll \begin{matrix} P_2^{(r)} \\ Q_2^{(r)} \end{matrix}(A_2, B_2, \Gamma, s) \quad \ldots \quad (10.32)$$

Now no term $\alpha_2{}^a\beta_2{}^b(\gamma e^{\sigma t})^n(\sigma-\nu)^c$ occurs in both $p_2{}^{(r)}$ and $u_2{}^{(r)}$, or in both $q_2{}^{(r)}$ and $v_2{}^{(r)}$, for by definition $p_2{}^{(r)}$, $q_2{}^{(r)}$ contain only those terms for which $n\neq 0$ and $u_2{}^{(r)}$, $v_2{}^{(r)}$ only those terms for which $n=0$, hence from (10.32) we may infer that

$$\begin{matrix} p_2{}^{(r)} \\ q_2{}^{(r)} \end{matrix}(\alpha_2,\ \beta_2,\ \gamma e^{\sigma t},\ \sigma-\nu) \ll \begin{matrix} \mathrm{P}_2{}^{(r)} \\ \mathrm{Q}_2{}^{(r)} \end{matrix}(\mathrm{A}_2,\ \mathrm{B}_2,\ \Gamma,\ s) \quad \ldots \ldots \quad (10.33)$$

$$\begin{matrix} u_2{}^{(r)} \\ v_2{}^{(r)} \end{matrix}(\alpha_2,\ \beta_2,\ \gamma e^{\sigma t},\ \sigma-\nu) \ll \begin{matrix} \mathrm{P}_2{}^{(r)} \\ \mathrm{Q}_2{}^{(r)} \end{matrix}(\mathrm{A}_2,\ \mathrm{B}_2,\ \Gamma,\ s). \quad \ldots \ldots \quad (10.34)$$

Now the terms of $\zeta_1{}^{(r)}$, $p_1{}^{(r)}$ are in one-one correspondence, any term of $\zeta_1{}^{(r)}$ being derived from the corresponding term of $p_1{}^{(r)}$ by dividing it by $(n\sigma-\rho)$ where n is integral. Since ρ is not a multiple of σ* the moduli of these divisors have a positive lower bound $\kappa_1|\sigma|$, where κ_1 is the lower bound of $|(n\nu-\lambda)/\nu|$. Similarly we derive the terms of $\omega_1{}^{(r)}$ from those of $q_1{}^{(r)}$ by introducing divisors whose moduli are not less than $\kappa_2|\sigma|$, where κ_2 is the lower bound of $|(n\nu+\lambda)/\nu|$. The terms of $\zeta_2{}^{(r)}$ are derived from those of $p_2{}^{(r)}$ by introducing divisors of the form $n\sigma$ where $n\neq 0$, and of these the moduli are not less than $|\sigma|$; and $\omega_2{}^{(r)}$ is similarly derived from $a\zeta_2{}^{(r)}+q_2{}^{(r)}$. We take ε equal to the smallest of $\kappa_1|\sigma|$, $\kappa_2|\sigma|$, $|\sigma|$; then a comparison of the solutions of the first three of (10.29) with the solutions of the first three of (10.12) gives

$$\zeta_1{}^{(r)}(\alpha_2,\ \beta_2,\ \gamma e^{\sigma t},\ \sigma-\nu) \ll \mathrm{Z}_1{}^{(r)}(\mathrm{A}_2,\ \mathrm{B}_2,\ \Gamma,\ s),$$

$$\omega_1{}^{(r)}(\alpha_2,\ \beta_2,\ \gamma e^{\sigma t},\ \sigma-\nu) \ll \Omega_1{}^{(r)}(\mathrm{A}_2,\ \mathrm{B}_2,\ \Gamma,\ s),$$

$$\zeta_2{}^{(r)}(\alpha_2,\ \beta_2,\ \gamma e^{\sigma t},\ \sigma-\nu) \ll \mathrm{Z}_2{}^{(r)}(\mathrm{A}_2,\ \mathrm{B}_2,\ \Gamma,\ s).$$

From (10.33) and the last of these results we have

$$a\zeta_2{}^{(r)}(\alpha_2,\ \beta_2,\ \gamma e^{\sigma t},\ \sigma-\nu)+q_2{}^{(r)}(\alpha_2,\ \beta_2,\ \gamma e^{\sigma t},\ \sigma-\nu)$$
$$\ll \mathrm{AZ}_2{}^{(r)}(\mathrm{A}_2,\ \mathrm{B}_2,\ \Gamma,\ s)+\mathrm{Q}_2{}^{(r)}(\mathrm{A}_2,\ \mathrm{B}_2,\ \Gamma,\ s),$$

and hence comparing the solutions for $\omega_2{}^{(r)}$, $\Omega_2{}^{(r)}$,

$$\omega_2{}^{(r)}(\alpha_2,\ \beta_2,\ \gamma e^{\sigma t},\ \sigma-\nu) \ll \Omega_2{}^{(r)}(\mathrm{A}_2,\ \mathrm{B}_2,\ \Gamma,\ s).$$

Thus (10.30) are established for $i=r$ and are true for $i=2,\ 3,\ \ldots\ \infty$, provided they are true for $i=1$; this, however, is obvious, since

$$\zeta_1{}^{(1)}=\omega_1{}^{(1)}=0, \qquad \zeta_2{}^{(1)}=\alpha_2. \qquad \omega_2{}^{(1)}=\beta_2$$

$$\mathrm{Z}_1{}^{(1)}=\Omega_1{}^{(1)}=0, \qquad \mathrm{Z}_2{}^{(1)}=\mathrm{A}_2, \qquad \Omega_2{}^{(1)}=\mathrm{B}_2+\mathrm{AA}_2/\varepsilon.$$

Moreover, the satisfaction of the conditions (10.30) for $i=1,2,\ldots\infty$ has been shown

* We have $\rho/\sigma=\lambda/\nu$.

to imply the satisfaction of (10.34) for $i = 2, 3, \ldots \infty$. We infer, then, that if the series (10.24) are convergent for

$$A_2 = A_2^0 > 0, \quad B_2 = B_2^0 > 0, \quad s = s_0 > 0, \quad \Gamma = \Gamma_0 > 0,$$

then both the series (10.9) and the series (10.14) are absolutely convergent for

$$|\alpha_2| \leq A_2^0, \quad |\beta_2| \leq B_2^0, \quad |\sigma - \nu| \leq s_0, \quad |\gamma e^{\sigma t}| = \Gamma_0\,;$$

(we write $|\gamma e^{\sigma t}| = \Gamma_0$ and not $|\gamma e^{\sigma t}| \leq \Gamma_0$, because both positive and negative powers of $\gamma e^{\sigma t}$ occur).

It remains only to consider the convergence of the comparison series (10.24). For this purpose we take the Φ_k, Ψ_k of the form (10.21), so that the comparison equations (10.23) are

$$\varepsilon Z_1 = \varepsilon \Omega_1 = \varepsilon (Z_2 - A_2) = \varepsilon (\Omega_2 - B_2) - AZ_2$$
$$= L\left\{\frac{(Z_1 + \Omega_1 + Z_2 + \Omega_2)^2}{\delta^2} + \frac{s\,(Z_1 + \Omega_1 + Z_2 + \Omega_2)}{\delta}\right\} \Big/ \left(1 - \frac{Z_1 + \Omega_1 + Z_2 + \Omega_2}{\delta}\right),$$

where $L = C\Sigma\,(\Gamma/\delta_2)^n + B\Sigma\,(\delta_1/\Gamma)^n$, and is a Laurent series in Γ, absolutely convergent when $\delta_1 < |\Gamma| < \delta_2$. Write

$$S = Z_1 + \Omega_1 + Z_2 + \Omega_2, \qquad L' = L\,(4/\varepsilon + A/\varepsilon^2); \qquad \alpha = A_2 + B_2 + AA_2/\varepsilon\,;$$

then we find

$$(S - \alpha)\left(1 - \frac{S}{\delta}\right) = L'\left(\frac{S^2}{\delta^2} + \frac{sS}{\delta}\right),$$

and the solution for S which vanishes with A_2, B_2 and s, *i.e.*, with α, s, is

$$S = \frac{\delta\,(\delta + \alpha - sL') - \delta\,\{(\delta + \alpha - sL')^2 - 4\alpha\,(L' + \delta)\}^{\frac{1}{2}}}{2\,(L' + \delta)}.$$

Thus

$$Z_1 = \Omega_1 = Z_2 - A_2 = (\Omega_2 - B_2 - AA_2/\varepsilon)/(1 + A/\varepsilon)$$
$$= \frac{\delta^2 - \delta\alpha - 2L'\alpha - \delta sL' - \delta\,\{(\delta + \alpha - sL')^2 - 4\alpha\,(L' + \delta)\}^{\frac{1}{2}}}{2\,(L' + \delta)\,(4 + A/\varepsilon)}.$$

Here the numerator is expansible in a series of positive powers of

$$\alpha L'\,(L' + \delta)\,(\alpha + \delta s)/(\delta^2 - \delta\alpha - 2L'\alpha - \delta sL'),$$

and we find

$$Z_1 = \Omega_1 = \ldots = \frac{\varepsilon}{4\varepsilon + A}\left\{\frac{\alpha L'\,(\alpha + \delta s)}{\delta^2 - \delta\alpha - 2L'\alpha - \delta sL'} + \frac{(L' + \delta)\alpha^2 L'^2\,(\alpha + \delta s)^2}{(\delta^2 - \delta\alpha - 2L'\alpha - \delta sL')^3} + \ldots\right\},$$

the series on the right having all its coefficients positive. Finally the denominators

$(\delta^2-\delta\alpha-2L'\alpha-\delta sL')^{-n}$ are expansible in powers of $\left(\frac{\alpha}{\delta}+\frac{sL'}{\delta}+\frac{2L'\alpha}{\delta^2}\right)$ with positive coefficients, giving

$$Z_1=\Omega_1=\ldots=\frac{\alpha L(\alpha+\delta s)}{\varepsilon\delta^2}\left(1+\frac{\alpha}{\delta}+\frac{4sL}{\delta\varepsilon}+\frac{sLA}{\delta\varepsilon^2}+\frac{2\alpha AL}{\delta^2\varepsilon^2}+\frac{8\alpha L}{\delta^2\varepsilon}+\ldots\right), \quad (10.35)$$

and in this final series only positive powers of α, s, L appear. The solution (10.35) must be identical with the solution (10.24). The conditions for the convergence of these expansions are, for real positive values of α, δ, s, L′,

$$\left.\begin{array}{l}4\alpha L'(L'+\delta)(\alpha+\delta s)<(\delta^2-\delta\alpha-2L'\alpha-\delta sL')^2\\ \delta\alpha+2L'\alpha+\delta sL'<\delta^2\end{array}\right\},$$

which are satisfied when

$$\alpha/\delta+\alpha L'/\delta^2<\tfrac{1}{4}, \qquad sL'/\delta+\alpha L'/\delta^2<\tfrac{1}{4}.$$

Inserting the values of α, L′ these conditions are, for positive values of A_2, B_2, s, L,

$$\left.\begin{array}{l}A_2+B_2+\dfrac{AA_2}{\varepsilon}<\dfrac{\delta}{4\{1+L(4\varepsilon+A)/\delta\varepsilon^2\}}\\ s+\dfrac{1}{\delta}\left(A_2+B_2+\dfrac{AA_2}{\varepsilon}\right)<\dfrac{\delta\varepsilon^2}{4L(4\varepsilon+A)}\end{array}\right\}. \quad (10.36)$$

Now the Laurent series L is uniformly convergent for $\Gamma_1\leq\Gamma\leq\Gamma_2$, provided $\delta_1<\Gamma_1<\Gamma_2<\delta_2$; hence for all (real positive) values of Γ within this range its sum does not exceed a certain finite positive quantity L_0. The positive powers of L occurring in (10.35) are likewise Laurent series in Γ whose sums for $\Gamma_1\leq\Gamma\leq\Gamma_2$ do not exceed the appropriate power of L_0. If, then, A_2^0, B_2^0, s_0 are positive values of A_2, B_2, s satisfying (10.36) when on the right we put $L=L_0$, the series (10.24) are uniformly convergent for

$$0\leq A_2\leq A_2^0, \quad 0\leq B_2\leq B_2^0, \quad 0\leq s\leq s_0, \quad \Gamma_1\leq\Gamma\leq\Gamma_2.$$

Hence by what has been already seen the series (10.9), (10.14) are absolutely and uniformly convergent for

$$|\alpha_2|\leq A_2^0, \quad |\beta_2|\leq B_2^0, \quad |\sigma-\nu|\leq s_0, \quad \Gamma_1\leq|\gamma e^{\sigma t}|\leq\Gamma_2.$$

Concerning this range of convergence it is to be noted: (i) the nearer we take Γ_1, Γ_2 to δ_1, δ_2 respectively, the larger will be L_0, and hence the smaller we shall have to take s_0, A_2^0, B_2^0. (ii) The values A_2^0, B_2^0, s_0 satisfying (10.36) tend to zero with ε, which is the smallest of $|\sigma|$, $|\sigma|.|n-\lambda/\nu|$, $|\sigma|.|n+\lambda/\nu|$ for integral values of n, so that $\varepsilon\to 0$ as λ/ν tends to an integral value.

To obtain from (10.5), (10.9) the periodic solutions of Family I we put $\beta_2=0$ and give α_2 a value satisfying (10.14); this value vanishes with $(\sigma-\nu)$, and is in general a power series in $(\sigma-\nu)$, and since it is derived from (10.14) by algebraic processes the series in question will be absolutely and uniformly convergent when $|\sigma-\nu|$ is sufficiently small. Thus the resulting series (10.9) in $\gamma e^{\sigma t}$, $\sigma - \nu$ are absolutely and uniformly convergent when $\Gamma_1 \leq |\gamma e^{\sigma t}| \leq \Gamma_2$ and $|\sigma - \nu|$ is sufficiently small, so they may be re-arranged in any manner; in particular we may write them as convergent Laurent series in $\gamma e^{\sigma t}$ with coefficients which are convergent power series (in general) in $(\sigma - \nu)$. The series which result from substituting (10.9) in (10.5) are similarly absolutely and uniformly convergent when $\Gamma_1 \leq |\gamma e^{\sigma t}| \leq \Gamma_2$ and $|\sigma - \nu|$ is sufficiently small.

Convergence of the Series giving Families II, III.—The series (10.9) are now constructed as power series in $\alpha_1 e^{\rho t}$, $\beta_1 e^{-\rho t}$, α_2, β_2, $(\sigma - \nu)$, the coefficients being Laurent series in $\gamma e^{\sigma t}$, where $\rho/\sigma = \lambda_0/\nu_0$, the ratio of two integers. The denominators introduced in their solution are of the form $(m\rho + n\sigma)$, where m, n are integers such that this quantity does not vanish; if $\rho = k\lambda_0$, $\sigma = k\nu_0$, their moduli have the lower bound $|k|$. The convergence of the series is proved exactly as above by taking as comparison equations

$$\left.\begin{aligned} \varepsilon Z_1 &= \varepsilon A_1 + \Phi_1(Z_k, \Omega_k, \Gamma, s), \qquad \varepsilon\Omega_1 = \varepsilon B_1 + \Psi_1(Z_k, \Omega_k, \Gamma, s) \\ \varepsilon Z_2 &= \varepsilon A_2 + \Phi_2(Z_k, \Omega_k, \Gamma, s), \qquad \varepsilon\Omega_2 = \varepsilon B_2 + AZ_2 + \Psi_2(Z_k, \Omega_k, \Gamma, s) \end{aligned}\right\}$$

with $\varepsilon = |k|$. The ranges of convergence established for $|\alpha_1 e^{\rho t}|$, $|\beta_1 e^{-\rho t}|$, $|\alpha_2|$, $|\beta_2|$, $|\sigma - \nu|$ tend to zero with $|k|$. The vanishing of the four convergent series of critical terms is secured by putting $\beta_2 = 0$ and α_1, β_1, α_2 equal to convergent power series in $(\sigma - \nu)^{\frac{1}{2}}$ (in general); and the series derived by inserting these values in (10.9) are absolutely convergent when $\Gamma_1 \leq |\gamma e^{\sigma t}| \leq \Gamma_2$ and $|\sigma - \nu|$ is sufficiently small, and may be re-arranged as Laurent series in $\gamma e^{\sigma t}$. That the range of convergence should tend to zero with $|k|$ is exactly what we should expect, since Families II, III do not exist when λ, ν are incommensurable.

§ 11. *Transformation of the Variational Equations to Normal Form.*

We must now show how to form the normalising transformation appropriate to any one of the periodic solutions as found in the last section, and also show that the series involved are convergent. If we take the variational equations formed from such a periodic solution as generating solution, we require a contact transformation which throws them into the "normal" form (5.6). For definiteness we take the general case in which $\lambda \neq 0$, $\lambda \neq \frac{1}{2}\nu$, $a \neq 0$, and suppose that the generating solution belongs to Family I. It is most convenient to start from the equations (10.8)—which are derived from (10.1) by the transformation (10.5)—so the generating solution is

$$\zeta_k = f_k(\gamma e^{\sigma t}, \sigma - \nu), \quad \omega_k = g_k(\gamma e^{\sigma t}, \sigma - \nu), \quad \ldots \ldots \quad (11.1)$$

where the f_k, g_k are power series in $(\sigma - \nu)$, vanishing with $(\sigma - \nu)$ and having coefficients which are Laurent series in $\gamma e^{\sigma t}$. To form the variational equations put in (10.8)

$$\zeta_k = f_k(\gamma e^{\sigma t}, \sigma - \nu) + \xi_k, \quad \omega_k = g_k(\gamma e^{\sigma t}, \sigma - \nu) + \eta_k, \quad \ldots \ldots \quad (11.2)$$

and neglect powers of the ξ_k, η_k above the first. When $\sigma = \nu$, (11.2) becomes $\zeta_k = \xi_k$, $\omega_k = \eta_k$, so the variational equations have the form of the linear terms in (10.8), viz.:

$$\frac{d\xi_1}{dt} = \rho\xi_1, \qquad \frac{d\eta_1}{dt} = -\rho\eta_1, \qquad \frac{d\xi_2}{dt} = 0, \qquad \frac{d\eta_2}{dt} = a\xi_2 \qquad (\rho = \lambda);$$

when $\sigma \neq \nu$ the variational equations are thus

$$\left.\begin{aligned} \frac{d\xi_1}{dt} &= \xi_1(\rho + a_{11}) + \eta_1 a_{12} + \xi_2 a_{13} + \eta_2 a_{14} \\ \frac{d\eta_1}{dt} &= \xi_1 a_{21} + \eta_1(-\rho + a_{22}) + \xi_2 a_{23} + \eta_2 a_{24} \\ \frac{d\xi_2}{dt} &= \xi_1 a_{31} + \eta_1 a_{32} + \xi_2 a_{33} + \eta_2 a_{34} \\ \frac{d\eta_2}{dt} &= \xi_1 a_{41} + \eta_1 a_{42} + \xi_2(a + a_{43}) + \eta_2 a_{44} \end{aligned}\right\}, \qquad (11.3)$$

where the a_{rs} are power series in $(\sigma - \nu)$ vanishing with $(\sigma - \nu)$, the coefficients being Laurent series in $\gamma e^{\sigma t}$. These must, of course, form a Hamiltonian set, so that, for example, $a_{11} = -a_{22}$, and we remember that $\rho/\sigma = \lambda/\nu$.

We search for a linear combination

$$z = \xi_1 w_1 + \eta_1 w_2 + \xi_2 w_3 + \eta_2 w_4, \qquad (11.4)$$

where the w_k are periodic functions of t (parameter σ), such that

$$\frac{dz}{dt} = \mu z, \qquad (11.5)$$

where μ is a constant at present undetermined, and is in fact an exponent of the periodic solution (11.1). It is easily shown that the w_k must satisfy the equations

$$\left.\begin{aligned} \frac{dw_1}{dt} &= w_1(\mu - \rho - a_{11}) - w_2 a_{21} - w_3 a_{31} - w_4 a_{41} \\ \frac{dw_2}{dt} &= -w_1 a_{12} + w_2(\mu + \rho - a_{22}) - w_3 a_{32} - w_4 a_{42} \\ \frac{dw_3}{dt} &= -w_1 a_{13} - w_2 a_{23} + w_3(\mu - a_{33}) - w_4(a + a_{43}) \\ \frac{dw_4}{dt} &= -w_1 a_{14} - w_2 a_{24} - w_3 a_{34} + w_4(\mu - a_{44}) \end{aligned}\right\}. \qquad (11.6)$$

Thus to a first approximation

$$\frac{dw_1}{dt} = (\mu - \rho)w_1, \qquad \frac{dw_2}{dt} = (\mu + \rho)w_2, \qquad \frac{dw_3}{dt} = \mu w_3 - a w_4, \qquad \frac{dw_4}{dt} = \mu w_4,$$

giving the general solution

$$w_1 = \alpha_1 e^{(\mu-\rho)t}, \qquad w_2 = \alpha_2 e^{(\mu+\rho)t}, \qquad w_3 = \alpha_3 e^{\mu t} - a\alpha_4 t e^{\mu t}, \qquad w_4 = \alpha_4 e^{\mu t},$$

where the α_k are arbitrary constants. Solutions for the w_k having the required periodicity exist if $\mu = 0$, ρ or $-\rho$, and are as follows:

$$\text{(i)}\ \mu = 0, \qquad w_1 = w_2 = w_4 = 0, \qquad w_3 = 1\,; \quad \ldots \quad (11.7)$$

$$\text{(ii)}\ \mu = \rho, \qquad w_1 = 1, \qquad w_2 = w_3 = w_3 = 0\,; \quad \ldots \quad (11.8)$$

$$\text{(iii)}\ \mu = -\rho, \quad w_2 = 1, \qquad w_1 = w_3 = w_4 = 0\,;$$

any other possible choice of μ, *e.g.*, $\mu = \sigma$, will be equivalent to one of the above, since the exponents of the solution (11.1) are by their nature indeterminate to an integral multiple of σ.

We now show how to obtain the solution of (11.6) to which (11.8) is the first approximation.

Put in (11.6)

$$\mu = \rho + \mu_1 \quad \ldots \quad (11.9)$$

$$\left.\begin{aligned} w_1 &= 1 + w_1^{(1)} + w_1^{(2)} + \ldots \\ w_k &= w_k^{(1)} + w_k^{(2)} + \ldots \quad (k = 2, 3, 4) \end{aligned}\right\} \quad \ldots \quad (11.10)$$

and separate out terms of the same order, regarding μ_1 and $(\sigma - \nu)$, as of order 1 and $w_k^{(r)}$ as of order r. We obtain

$$\left.\begin{aligned} \frac{dw_1^{(1)}}{dt} &= \mu_1 - (\sigma - \nu)\, b_{11} \\ \frac{dw_2^{(1)}}{dt} - 2\rho w_2^{(1)} &= -(\sigma - \nu)\, b_{12} \\ \frac{dw_3^{(1)}}{dt} - \rho w_3^{(1)} &= -(\sigma - \nu)\, b_{13} - a w_4 \\ \frac{dw_4^{(1)}}{dt} - \rho w_4^{(1)} &= -(\sigma - \nu)\, b_{14} \end{aligned}\right\},$$

where b_{rs} is the coefficient of $(\sigma - \nu)$ in a_{rs}, and is a Laurent series in $\gamma e^{\sigma t}$; and in general

$$\left.\begin{aligned} \frac{dw_1^{(r)}}{dt} &= f_1^{(r)}(w_k^{(1)}, \ldots \quad w_k^{(r-1)}, \mu_1, \sigma - \nu, \gamma e^{\sigma t}) \\ \frac{dw_2^{(r)}}{dt} - 2\rho w_2^{(r)} &= f_2^{(r)}(w_k^{(1)}, \ldots \quad w_k^{(r-1)}, \mu_1, \sigma - \nu, \gamma e^{\sigma t}) \\ \frac{dw_3^{(r)}}{dt} - \rho w_3^{(r)} &= -a w_4^{(r)} + f_3^{(r)}(w_k^{(1)}, \ldots \quad w_k^{(r-1)}, \mu_1, \sigma - \nu, \gamma e^{\sigma t}) \\ \frac{dw_4^{(r)}}{dt} - \rho w_4^{(r)} &= f_4^{(r)}(w_k^{(1)}, \ldots \quad w_k^{(r-1)}, \mu_1, \sigma - \nu, \gamma e^{\sigma t}) \end{aligned}\right\}, \quad . \quad (11.11)$$

where the $f_k^{(i)}$ are polynomials of order r, linear in the $w_k^{(i)}$.

Suppose the $w_k^{(1)}, \ldots w_k^{(r-1)}$ have been obtained as polynomials of orders $1, \ldots (r-1)$ in $(\sigma - \nu)$, μ_1 whose coefficients are Laurent series in $\gamma e^{\sigma t}$; then the right-hand sides of (11.11) become homogeneous polynomials of order r in $(\sigma - \nu)$, μ_1 whose coefficients are Laurent series in $\gamma e^{\sigma t}$, and a solution for the $w_k^{(r)}$ is obtained by adding the "particular integral" solutions corresponding to each term. As in § 10, a term $\mathrm{A}e^{n\sigma t}$ for which the particular integral is of the form $\mathrm{B}te^{n\sigma t}$, is called a *critical* term, and to obtain a solution of the desired form for the $w_k^{(r)}$ we omit the critical terms before integrating the equations. Since neither ρ nor 2ρ is an integral multiple of σ there will be critical terms only in the equation for $w_1^{(r)}$, and here the critical terms are those of the form $\mathrm{A}e^{n\sigma t}$ in which $n = 0$; for example, in the equation for $w_1^{(1)}$ the critical terms are

$$\mu_1 - (\sigma - \nu)\,\bar{b}_{11},$$

where $\bar{b}_{11}$ is the absolute term in the Laurent series b_{11}.

The series (11.10) as thus constructed satisfy the equations (11.6), provided μ_1 has such a value that the sum of the critical terms arising from the equations for $w_1^{(1)}, w_1^{(2)}, \ldots$ vanishes; writing C_r for the critical terms* in the equation for $w_1^{(r)}$, this condition is

$$\mu_1 - (\sigma - \nu)\,\bar{b}_{11} + \mathrm{C}_2 + \mathrm{C}_3 + \ldots = 0 \qquad \ldots\ldots \quad (11.12)$$

and it is soluble for μ_1 as a power series in $(\sigma - \nu)$. This series and the series (11.10) will shortly be proved convergent when $|\sigma - \nu|$ is sufficiently small.

The solution of (11.6) to which (11.7) is a first approximation may be constructed similarly, or may be obtained, by algebraic processes only, from the Hamiltonian function F as follows: by means of (10.5) and (11.2) we express F as a power series in the ξ_k, η_k; then, since F is an integral of the equations (10.1), the linear terms in this series, say $\mathrm{L}, \equiv \xi_1\psi_1 + \eta_1\psi_2 + \xi_2\psi_3 + \eta_2\psi_4$, give an integral of the variational equations (11.3), so that $d\mathrm{L}/dt = 0$ in virtue of (11.3). Evidently $w_k = \psi_k$ is the periodic solution of (11.6) belonging to the exponent $\mu = 0$.

Since F, when expressed in terms of the ζ_k, ω_k, gives a series convergent under the conditions (10.19), the series $\psi_1, \ldots \psi_4$ are absolutely and uniformly convergent under conditions of the form

$$\delta_1 \leq |\gamma e^{\sigma t}| \leq \delta_2, \qquad |\sigma - \nu| \leq s_0.$$

To prove the series (11.10), (11.12) convergent we use a method similar to that of § 10. The series on the right of (10.8) are absolutely convergent in the range (10.19), and in the transformation (11.2) the series f_k, g_k vanish with $(\sigma - \nu)$, and are absolutely convergent for

$$|\sigma - \nu| \leq c, \qquad \Gamma_1 \leq |\gamma e^{\sigma t}| \leq \Gamma_2,$$

* C_r is a homogeneous polynomial in $(\sigma - \nu)$, μ_1 of order r.

where c is a certain positive constant and (Γ_1, Γ_2) is any range interior to (δ_1, δ_2). Hence, in the equations (11.3), (11.6), the series a_{rs} are absolutely convergent for

$$|\sigma - \nu| \leq d, \qquad \Gamma_1 \leq |\gamma e^{\sigma t}| \leq \Gamma_2,$$

where d is a certain positive constant. As comparison equations we take

$$\left.\begin{aligned} \varepsilon W_1 &= \varepsilon + W_1(m + A_{11}) + W_2 A_{21} + W_3 A_{31} + W_4 A_{41} \\ \varepsilon W_2 &= W_1 A_{12} + W_2(m + A_{22}) + W_3 A_{32} + W_4 A_{42} \\ \varepsilon W_3 &= W_1 A_{13} + W_2 A_{23} + W_3(m + A_{33}) + W_4(A + A_{43}) \\ \varepsilon W_4 &= W_1 A_{14} + W_2 A_{24} + W_3 A_{34} + W_4(m + A_{44}) \end{aligned}\right\}, \quad (11.13)$$

where $A = |a|$ and the A_{rs} are series in arguments Γ, s, vanishing with s and such that

$$a_{rs}(\gamma e^{\sigma t}, \quad \sigma - \nu) \ll A_{rs}(\Gamma, s);$$

the arguments m, W_k, Γ, s correspond respectively to μ_1, w_k, $\gamma e^{\sigma t}$, $(\sigma - \nu)$, and ε is a constant whose value will be specified presently. These are to be solved for the W_k as series of positive powers of m, s, and positive and negative powers of Γ.

When $m = s = 0$ (11.13) give

$$W_1 = 1, \qquad W_2 = W_3 = W_4 = 0.$$

Substituting then

$$\left.\begin{aligned} W_1 &= 1 + W_1^{(1)} + W_1^{(2)} + \dots \\ W_k &= W_k^{(1)} + W_k^{(2)} + \dots \qquad (k = 2, 3, 4) \end{aligned}\right\}, \quad \dots \quad (11.14)$$

we obtain on separating out terms of the same order

$$\varepsilon W_1^{(1)} = m + sB_{11}, \quad \varepsilon W_2^{(1)} = sB_{12}, \quad \varepsilon W_3^{(1)} = sB_{13} + AW_4^{(1)}, \quad \varepsilon W_4^{(1)} = sB_{14},$$

where B_{rs} is the coefficient of s in A_{rs}; and in general

$$\left.\begin{aligned} \varepsilon W_1^{(r)} &= F_1^{(r)}(W_k^{(1)}, \dots W_k^{(r-1)}, m, s, \Gamma) \\ \varepsilon W_2^{(r)} &= F_2^{(r)}(W_k^{(1)}, \dots W_k^{(r-1)}, m, s, \Gamma) \\ \varepsilon W_3^{(r)} &= AW_4^{(r)} + F_3^{(r)}(W_k^{(1)}, \dots W_k^{(r-1)}, m, s, \Gamma) \\ \varepsilon W_4^{(r)} &= F_4^{(r)}(W_k^{(1)}, \dots W_k^{(r-1)}, m, s, \Gamma) \end{aligned}\right\},$$

where

$$f_i^{(r)}(w_k^{(1)}, \dots w_k^{(r-1)}, \mu_1, \sigma - \nu, \gamma e^{\sigma t}) \ll F_i^{(r)}(W_k^{(1)}, \dots W_k^{(r-1)}, m, s, \Gamma).$$

From any term $\mu_1^{n_1}(\sigma - \nu)^{n_2}(\gamma e^{\sigma t})^{n_3}$ on the right of (11.11) we obtain the corresponding "particular integral" term by dividing by an expression of one of the forms

$$n_3\sigma, \quad n_3\sigma - 2\rho, \quad n_3\sigma - \rho;$$

since neither ρ nor 2ρ is a multiple of σ the moduli of these divisors have a positive lower bound, and to this we equate ε. We may then show, as in § 10, that

$$w_k^{(r)}(\mu_1, \sigma - \nu, \gamma e^{\sigma t}) \leqslant W_k^{(r)}(m, s, \Gamma) \quad (r = 1, 2, \ldots \infty),$$

and that

$$C_r(\mu_1, \sigma - \nu, \gamma e^{\sigma t}) \leqslant \varepsilon W_1^{(r)}(m, s, \Gamma) \quad (r = 1, 2, \ldots \infty).$$

The comparison series (11.14) are easily shown to be convergent when $\Gamma_1 \leq \Gamma \leq \Gamma_2$ for sufficiently small positive values of m, s, and hence the series (11.10), (11.12) are absolutely and uniformly convergent under conditions of the form

$$\Gamma_1 \leq |\gamma e^{\sigma t}| \leq \Gamma_2, \qquad |\sigma - \nu| \leq s_0, \qquad |\mu_1| \leq \mu_1^0.$$

We may solve (11.12) for μ_1 as an absolutely convergent power series in $(\sigma - \nu)$ vanishing with $(\sigma - \nu)$, and (11.10) then become series in $(\sigma - \nu)$, $\gamma e^{\sigma t}$ absolutely and uniformly convergent under conditions of the form

$$\Gamma_1 \leq |\gamma e^{\sigma t}| \leq \Gamma_2, \qquad |\sigma - \nu| \leq s_1. \qquad (11.15)$$

We have now shown how to find linear combinations z_1, z_2 of the ξ_k, η_k with periodic coefficients (parameter σ), say

$$\left.\begin{aligned} z_1 &= \xi_1 w'_1 + \eta_1 w'_2 + \xi_2 w'_3 + \eta_2 w'_4 \\ z_2 &= \xi_1 w''_1 + \eta_1 w''_2 + \xi_2 w''_3 + \eta_2 w''_4 \end{aligned}\right\}, \qquad (11.16)$$

satisfying respectively the equations

$$\frac{dz_1}{dt} = \mu z_1, \qquad \frac{dz_2}{dt} = 0,$$

Here μ is a convergent power series in $(\sigma - \nu)$, and the w'_k, w''_k are series in $(\sigma - \nu)$, $\gamma e^{\sigma t}$ absolutely and uniformly convergent under conditions of the form (11.15); and when $\sigma - \nu = 0$ we have

$$w'_1 = 1, \quad w'_2 = w'_3 = w'_4 = 0, \quad \mu = \lambda \quad (cf.\ 11.8);$$
$$w''_3 = 1, \quad w''_1 = w''_2 = w''_4 = 0 \qquad (cf.\ 11.7).$$

Thus, when $|\sigma - \nu|$ is sufficiently small, we have $\mu \not\equiv 0$ (mod. σ), and so*

$$(z_1, z_2) = 0.$$

Then if we put

$$\left.\begin{aligned} (w'_1 w''_3 - w''_1 w'_3)\, u_1 &= w''_3 \eta_1 - w''_1 \eta_2 + Az_1 + Bz_2 \\ (w_1' w''_3 - w''_1 w_3')\, u_2 &= -w'_3 \eta_1 + w'_1 \eta_2 + Bz_1 + Cz_2 \end{aligned}\right\}, \qquad (11.17)$$

* Cherry, 'Proc. Lond. Math. Soc.,' vol. 26, p. 211 (1927), "On the transformation of Hamiltonian systems of linear differential equations with constant or periodic coefficients," § 10.

the transformation from the ξ_k, η_k to the z_k, u_k specified by (11.16), (11.17) is a contact transformation, and A, B, C can be so determined as series in $(\sigma-\nu)$, $\gamma e^{\sigma t}$ that the equations (11.3) become under this transformation

$$\frac{dz_1}{dt}=\mu z_1, \qquad \frac{du_1}{dt}=-\mu u_1, \qquad \frac{dz_2}{dt}=0, \qquad \frac{du_2}{dt}=cz_2, \quad . \quad . \quad . \quad (11.18)$$

where c is a constant.* Since

$$w'_1 w''_3 - w''_1 w'_3 = 1 + (\text{terms vanishing with } (\sigma-\nu))$$

we may solve (11.16), (11.17) for the ξ_k, η_k as linear functions of the z_k, u_k with coefficients which are series of positive powers of $(\sigma-\nu)$ and positive and negative powers of $\gamma e^{\sigma t}$, and it is a trivial matter to prove that these series are absolutely and uniformly convergent under conditions of the form (11.15). Call this transformation

$$\xi_k = \theta'_k(z_r, u_r, \gamma e^{\sigma t}, \sigma-\nu), \qquad \eta_k = \chi'_k(z_r, u_r, \gamma e^{\sigma t}, \sigma-\nu); \quad . \quad . \quad (11.19)$$

then by combining (10.5), (11.2), (11.19) *we obtain the "normalising" linear contact transformation appropriate to a generating solution of parameter* σ *belonging to Family I, and the series here involved are absolutely convergent when* $\Gamma_1 \leq |\gamma e^{\sigma t}| \leq \Gamma_2$ *and* $|\sigma-\nu|$ *is sufficiently small.*

Now when F is expressed in terms of the ζ_k, ω_k by means of (10.5) we obtain by hypothesis a series absolutely convergent when $\delta_1 \leq |\gamma e^{\sigma t}| \leq \delta_2$ and $|\zeta_k|$, $|\omega_k|$ are sufficiently small. By means of (11.2), (11.19) we may express F as a power series in the z_k, u_k. Since the ζ_k, ω_k vanish with $(\sigma-\nu)$ and the z_k, u_k, and the series involved in the transformations (11.2), (11.19) are uniformly convergent, we may make the $|\zeta_k|$, $|\omega_k|$ arbitrarily small by taking $|\sigma-\nu|$, $|z_k|$, $|u_k|$ sufficiently small. Using this fact, together with the absolute convergence of the transformations, we see that F becomes a power series in the z_k, u_k with coefficients which are series in $(\sigma-\nu)$, $\gamma e^{\sigma t}$, absolutely convergent when $\Gamma_1 \leq |\gamma e^{\sigma t}| \leq \Gamma_2$ and $|\sigma-\nu|$, $|z_k|$, $|u_k|$ are sufficiently small. *Thus* F *is regular*† *in the neighbourhood of any periodic solution of Family I sufficiently near the generating solution.*

By using similar methods we may construct the normalising contact transformation appropriate to a generating solution of Family II or III, and that appropriate to a periodic solution of Family I or II when $a=0$ or $\lambda=0$ or $\lambda=\frac{1}{2}\nu$, and we may similarly establish the convergence of the series here involved and the regularity of F in the neighbourhood of such a periodic solution.

Thus we have shown that in the neighbourhood of any REGULAR† *periodic solution the families furnished by the formal process of* §§ 5–9 *consist of* REGULAR *periodic solutions.*

* CHERRY, *loc. cit.*, § 12.

† For explanation of this term see § 10, p. 185 above.

Periodic Solutions in the Neighbourhood of an Equilibrium Solution.

§ 12.

An equilibrium solution of the equations (5.1) is a solution of the form

$$x_k = x_k^0, \qquad y_k = y_k^0, \quad \ldots \ldots \ldots \ldots \quad (12.1)$$

where the x_k^0, y_k^0 are constants; it is, of course, a special case of a periodic solution. The condition that (12.1) should be an equilibrium solution is that the right-hand sides of the equations (5.1) should vanish at this point. We suppose that F is expansible in powers of the $x_k - x_k^0$, $y_k - y_k^0$.

The theory of periodic solutions in the neighbourhood of such a point follows in its main lines the procedure of §§ 5–11, but differs so much in its details as to require a separate exposition. Lack of space forbids more than the mention of a few of the salient points and a statement of the results.

A normalising transformation in which $x_k - x_k^0$, $y_k - y_k^0$ are homogeneous linear functions of the ζ_k, ω_k, with constant coefficients, reduces the equations (5.1) to the form (*cf.* (5.10))

$$\frac{d\zeta_k}{dt} = \frac{\partial G}{\partial \omega_k}, \qquad \frac{d\omega_k}{dt} = -\frac{\partial G}{\partial \zeta_k}, \quad \ldots \ldots \ldots \quad (12.2)$$

in which G is a power series in the ζ_k, ω_k beginning with quadratic terms G_2; if the exponents of the solution (12.1) are $\pm\lambda_1$, $\pm\lambda_2$ we have

when $\lambda_1 \neq 0,\ \lambda_2 \neq 0,\ \lambda_1 \neq \lambda_2$, $\quad G_2 = \lambda_1\zeta_1\omega_1 + \lambda_2\zeta_2\omega_2$;

when $\lambda_1 = \lambda_2 \neq 0$, $\quad G_2 = \lambda_1(\zeta_1\omega_1 + \zeta_2\omega_2) + a\zeta_2\omega_1$;

when $\lambda_1 \neq 0,\ \lambda_2 = 0$, $\quad G_2 = \lambda_1\zeta_1\omega_1 + \tfrac{1}{2}a\zeta_2^2$;

when $\lambda_1 = \lambda_2 = 0$, $\quad G_2 = a\zeta_2\omega_1 + \tfrac{1}{2}b\zeta_1^2 + c\zeta_1\zeta_2 + \tfrac{1}{2}d\zeta_2^2$.

The equations (12.2) may be formally solved by power series in the arguments

$$\alpha_1 e^{\lambda_1 t}, \quad \beta_1 e^{-\lambda_1 t}, \qquad \alpha_2 e^{\lambda_2 t}, \quad \beta_2 e^{-\lambda_2 t},$$

with coefficients polynomial in t, and the Lagrange brackets of the integration-constants α_k, β_k have the values (5.4). Putting $t = 0$ in this general solution and replacing α_k, β_k by z_k, u_k respectively, we obtain a formal contact transformation

$$x_k = f_k(z_r, u_r), \quad y_k = g_k(z_r, u_r), \quad \ldots \ldots \ldots \quad (12.3)$$

whereby the equations (5.1) become (*cf.* (5.21))

$$\frac{dz_k}{dt} = \frac{\partial K}{\partial u_k}, \qquad \frac{du_k}{dt} = -\frac{\partial K}{\partial z_k}, \quad \ldots \ldots \ldots \quad (12.4)$$

where K is a power series:

$$K \equiv \Sigma A_{a_1 b_1 a_2 b_2} z_1^{a_1} u_1^{b_1} z_2^{a_2} u_2^{b_2},$$

in which for each term the indices satisfy the relation

$$\lambda_1 (a_1 - b_1) + \lambda_2 (a_2 - b_2) = 0.$$

Thus (i) if λ_1, λ_2 are connected by no relation of the form

$$A_1 \lambda_1 = A_2 \lambda_2, \quad \ldots \ldots \ldots \ldots \quad (12.5)$$

where A_1, A_2 are integers not both zero, K is a power series in two arguments $v_1, = z_1 u_1$ and $v_2, = z_2 u_2$; (ii) if there is a relation (12.5) we may suppose A_1, A_2 mutually prime, and K is a power series in the four arguments

$$v_1, = z_1 u_1\ ;\ v_2, = z_2 u_2\ ;\ w, = z_1^{A_1} u_2^{A_2}\ ;\ w', = u_1^{A_1} z_2^{A_2}$$

which are connected by the relation $ww' = v_1^{A_1} v_2^{A_2}$; (iii) if $\lambda_1 = \lambda_2 = 0$, K is a power series in the four arguments z_1, u_1, z_2, u_2. We see that if either of the exponents λ_1, λ_2 is taken to correspond with the λ of § 5, the other corresponds in a sense to ν.

From any formally periodic solution of (12.4) we can deduce through (12.3) a formally periodic solution of (5.1). Solutions so derived may be shown to be regular by a method similar to that of § 10, *under the sole hypothesis that the Hamiltonian function* F *is regular at the point* (x_k^0, y_k^0).

Case I: λ_1, λ_2 *incommensurable* (*cf.* § 6).—In (12.4) we have $K \equiv K(v_1, v_2)$ and the equations are

$$\frac{dz_k}{dt} = z_k \frac{\partial K}{\partial v_k}, \qquad \frac{du_k}{dt} = - u_k \frac{\partial K}{\partial v_k} \qquad (k = 1, 2)$$

possessing the integrals v_1, v_2. They possess the two families of periodic solutions

$$\text{(I)}\ z_1 = \alpha_1 e^{\sigma_1 t}, \qquad u_1 = \beta_1 e^{-\sigma_1 t}, \quad z_2 = u_2 = 0, \quad \sigma_1 = \left(\frac{\partial K}{\partial v_1}\right)_{v_1 = \alpha_1\beta_1,\ v_2 = 0}; \quad (12.6)$$

$$\text{(II)}\ z_1 = u_1 = 0, \quad z_2 = \alpha_2 e^{\sigma_2 t}, \quad u_2 = \beta_2 e^{-\sigma_2 t}, \quad \sigma_2 = \left(\frac{\partial K}{\partial v_2}\right)_{v_1 = 0,\ v_2 = \alpha_2\beta_2};$$

each family depends on two arbitrary constants, *e.g.*, α_1, β_1 for Family I, but is to be regarded as only singly-infinite since an alteration in the two constants which leaves their product unchanged is equivalent to the addition of a constant to t. In general we may solve (12.6) for $\alpha_1\beta_1$ as a power series in $(\sigma_1 - \lambda_1)$ vanishing with $(\sigma_1 - \lambda_1)$, and then writing

$$\alpha_1 = (\alpha_1\beta_1)^{\frac{1}{2}}\gamma, \qquad \beta_1 = (\alpha_1\beta_1)^{\frac{1}{2}}\gamma^{-1}, \qquad \gamma = (\alpha_1/\beta_1)^{\frac{1}{2}},$$

Family I becomes

$$z_1 = \gamma e^{\sigma_1 t} P, \qquad u_1 = (\gamma e^{\sigma_1 t})^{-1} Q, \qquad z_2 = u_2 = 0, \quad \ldots\ldots \quad (12.7)$$

where P, Q are power series in $(\sigma_1 - \lambda_1)^{\frac{1}{2}}$ vanishing with $(\sigma_1 - \lambda_1)$; and similarly for Family II. Both families have the solution (12.1) as a limiting member.

When F is a real function of the x_k, y_k and the equilibrium solution (12.1) is real, both families of periodic solutions have real members; the real solutions of either family have their periods real only when the corresponding exponent λ_1 or λ_2 is pure-imaginary.

The non-zero exponents of a solution of Family I are in general expressible as power series in $(\sigma_1 - \lambda_1)$, reducing to $\pm \lambda_2$ when $\sigma_1 = \lambda_1$; and similarly for Family II.

CASE II: λ_1, λ_2 *commensurable* (*cf.* § 7).—Families I, II exist as before in general; we find also Families III, IV of solutions analogous to Families II, III of § 7, and so to POINCARÉ'S "solutions du deuxième genre"; the families in question may be thought of as branching from either Family I or Family II at the equilibrium solution (12.1). The solutions in question are of the form

$$z_1 = \alpha_1 e^{A_2\sigma t}, \qquad u_1 = \beta_1 e^{-A_2\sigma t}, \qquad z_2 = \alpha_2 e^{A_1\sigma t}, \qquad u_2 = \beta_2 e^{-A_1\sigma t}, \quad . \quad . \quad . \quad (12.8)$$

where the α_k, β_k are constants subject to the conditions of giving values of $v_1\ (= \alpha_1\beta_1)$, $v_2\ (= \alpha_2\beta_2)$, $w\ (= \alpha_1^{A_1}\beta_2^{A_2})$, $w'\ (= \beta_1^{A_1}\alpha_2^{A_2})$ which satisfy the equations (*cf.* (7.12), (7.14), (7.15)):

$$\left.\begin{aligned} ww' &= v_1^{A_1}v_2^{A_2} \\ w\frac{\partial K}{\partial w} &= w'\frac{\partial K}{\partial w'} \\ v_1v_2\left(A_1\frac{\partial K}{\partial v_1} - A_2\frac{\partial K}{\partial v_2}\right) + w\frac{\partial K}{\partial w}&(A_1^2v_2 - A_2^2v_1) = 0 \end{aligned}\right\}; \quad . \quad . \quad . \quad . \quad (12.9)$$

while σ is a constant given in terms of v_1, v_2, w, w' by

$$\sigma = \frac{1}{A_2}\frac{\partial K}{\partial v_1} + \frac{A_1}{A_2}\frac{w}{v_1}\frac{\partial K}{\partial w}.$$

These conditions lead in general to *two* distinct solutions for the α_k, β_k as power series in $(\sigma - \lambda_1/A_2)^{\frac{1}{2}}$ vanishing with $(\sigma - \lambda_1/A_2)$, and the solution (12.8) is of the form

$$Z_1 = (\gamma e^{\sigma t})^{A_2}P_1, \qquad u_1 = (\gamma e^{\sigma t})^{-A_2}Q_1, \qquad z_2 = (\gamma e^{\sigma t})^{A_1}P_2, \qquad u_2 = (\gamma e^{\sigma t})^{-A_1}Q_2,$$

where the P_k, Q_k are series of this nature and γ is an arbitrary constant.

When (i) F is a real function of the x_k, y_k, (ii) the equilibrium solution (12.1) is real, and (iii) λ_1, λ_2 are pure-imaginary, both Families I and II possess real periodic solutions of real period; Families III and IV, on the other hand, will possess such real solutions only if a certain relation of inequality between the coefficients of the terms of degrees 2, 3 and 4 in the power series G is satisfied.

For real solutions of Families I, II, the non-zero exponents are in general pure-imaginary; when Families III and IV possess real solutions their non-zero exponents are in general real for one family and pure-imaginary for the other.

The following table shows what is in general the nature of the non-zero exponents of real solutions of the various families when λ_1, λ_2 are pure-imaginary. An asterisk denotes that to the family belong real solutions if, and only if, the relation of inequality already mentioned holds. A dash in any column denotes that there is no family of the corresponding sort having the origin as a limiting member. The assignment of numbers to the families in those cases in which they are less than four in number is made so as to preserve the greatest possible analogy with the most general case in which four families exist, viz., λ_1/λ_2 rational but not integral.

Nature of the Exponents of Families of Real Periodic Solutions branching from an Equilibrium Solution whose Exponents $\pm\lambda_1$, $\pm\lambda_2$ *are Pure-imaginary* ($|\lambda_1| \geq |\lambda_2|$).

λ_1/λ_2.	Family I.	Family II.	Family III.	Family IV.
Not commensurable	Pure-imaginary	Pure-imaginary	—	—
Rational but not integral	Pure-imaginary	Pure-imaginary	* Pure-imaginary	* Real.
Integral, > 3	Pure-imaginary	Pure-imaginary	* Pure-imaginary	* Real.
$= 3$	Pure-imaginary	Pure-imaginary or real	* Pure-imaginary	* Real.
$= 2$	Complex	—	* Pure-imaginary	* Pure-imaginary.
$= 1$	—	—	* Pure-imaginary	* Real.
$= 0$	—	—	Pure-imaginary or real.	—

Suppose that we know an equilibrium solution (12.1) in whose neighbourhood F is regular. Let $\mathfrak{S}$ be a periodic solution in the neighbourhood of this equilibrium solution as furnished by the preceding theory. Then the method of § 10 may be used to give periodic solutions in the neighbourhood of $\mathfrak{S}$, provided that we know the normalising contact transformation appropriate to $\mathfrak{S}$. This contact transformation may be found, the series involved may be proved convergent, and F may be shown to be regular in the neighbourhood of $\mathfrak{S}$, by a method similar to that of § 11, so $\mathfrak{S}$ is a regular periodic solution. Call $\mathfrak{S}_1$ any periodic solution in the neighbourhood of $\mathfrak{S}$ as found by the method of § 10; we may take $\mathfrak{S}_1$ as generating solution and proceed by the same method to find regular periodic solutions in *its* neighbourhood, and so on. We see, therefore, the importance of the results of this section, for the knowledge merely of a regular *equilibrium* solution enables us to construct as many families of regular periodic solutions as we like.

The finding of an equilibrium solution is a problem of "algebraic" nature, for the equilibrium solutions are the sets of values (x_k, y_k) for which $\partial F/\partial x_k = \partial F/\partial y_k = 0$. We can, for instance, assert that if F is a polynomial in the x_k, y_k the equations (5.1) will certainly possess equilibrium solutions, and therefore also an infinite number of families of regular periodic solutions.

EQUATIONS DEPENDING ON AN ARBITRARY PARAMETER μ.

§ 13. *Variation of Periodic Solutions with an Arbitrary Parameter on which the Equations Depend.*

We shall now show that when the equations depend analytically on an arbitrary parameter μ, any family of regular periodic solutions varies in general continuously* with μ. One important consequence of this result is that we may assert that if a certain system of equations is known to possess periodic solutions, then all systems of equations which differ sufficiently little from the first system must possess periodic solutions also.

We take the equations

$$\frac{dx_k}{dt} = \frac{\partial (F + \mu H)}{\partial y_k}, \qquad \frac{dy_k}{dt} = -\frac{\partial (F + \mu H)}{\partial x_k} \quad (k = 1, 2), \quad . \quad . \quad . \quad (13.1)$$

where F is an analytic function of the x_k, y_k and H is an analytic function of μ and the x_k, y_k, and suppose that when $\mu = 0$ these equations possess the regular periodic solution

$$x_k = \phi_k (\gamma e^{\nu t}), \qquad y_k = \psi_k (\gamma e^{\nu t}), \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (13.2)$$

its exponents being $0, 0, \pm \lambda$. Putting $x_k = \phi_k + \xi_k$, $y_k = \psi_k + \eta_k$ the function F is supposed developable in powers of the ξ_k, η_k and H in powers of μ and the ξ_k, η_k. Then, by a method similar to that of § 5 *we may construct a contact transformation*

$$x_k = f_k (z_r, u_r, \mu, \gamma e^{\sigma t}), \qquad y_k = g_k (z_r, u_r, \mu, \gamma e^{\sigma t}), \quad . \quad . \quad . \quad . \quad . \quad . \quad (13.3)$$

where the f_k, g_k *are power series in* μ *and the* z_k, u_k *with coefficients which are Laurent series in* $\gamma e^{\sigma t}$, *under which the equations* (13.1) *become*

$$\frac{dz_k}{dt} = \frac{\partial K}{\partial u_k}, \qquad \frac{du_k}{dt} = -\frac{\partial K}{\partial z_k}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (13.4)$$

where K *is a series*:

$$K \equiv F + M + \mu H$$

$$\equiv \Sigma \left(A_{a_1 b_1 a_2 b_2 cd} + \sigma B_{a_1 b_1 a_2 b_2 cd}\right) z_1^{a_1} u_1^{b_1} z_2^{a_2} u_2^{b_2} (\gamma e^{\sigma t})^c \mu^d \ldots \quad . \quad . \quad . \quad . \quad . \quad (13.5)$$

in which for each term the indices are integers satisfying

$$\left.\begin{array}{c} \lambda_1 (a_1 - b_1) + \nu c = 0 \\ a_1 \geq 0, \quad b_1 \geq 0, \quad a_2 \geq 0, \quad b_2 \geq 0, \quad d \geq 0 \end{array}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (13.6)$$

We first apply the linear contact transformation (5.11), and then solve the resulting equations, treating μ as of the same order as the α_k, β_k; we thus obtain the successive terms in the series for the ζ_k, ω_k as homogeneous polynomials in μ, $\alpha_1 e^{\lambda t}$, $\beta_1 e^{-\lambda t}$, α_2, β_2 with coefficients which involve t polynomially and $\gamma e^{\nu t}$ through positive and negative powers. The dependence of the ζ_k, ω_k on the arbitrary μ makes no difference to the argument whereby we prove that the Lagrange brackets of the α_k, β_k have the values (5.14), or to the argument whereby we obtain the contact transformation (5.20) and establish the special form of the series for K.

* Here and below, functions which are stated to vary continuously with μ are actually analytic functions of μ, regular at the point $\mu = 0$.

From the relation (13.6) we may show, as in § 5, that

(i) when λ, ν are incommensurable, K involves z_1, u_1 only through their product v, and is independent of $\gamma e^{\sigma t}$;

(ii) when λ, ν are commensurable and $\lambda \neq 0$, K involves z_1, u_1, t only through the arguments

$$v = z_1 u_1 ; \qquad w_1 = z_1^{\nu_0} (\gamma e^{\sigma t})^{-\lambda_0} ; \qquad w', = u_1^{\nu_0} (\gamma e^{\sigma t})^{\lambda_0} ;$$

(iii) when $\lambda = 0$, K does not depend on the argument $\gamma e^{\sigma t}$. Of course, when $\mu = 0$, the equations (13.4) are identical with (5.21), so the terms in K which are independent of μ and are of lowest degree in $(\sigma - \nu)$ and the z_k, u_k have the form (5.25), (5.27) or (5.29), according to the value of λ/ν.

As in §§ 6, 7, we obtain periodic solutions of (13.1) through the transformation (13.3) by searching for solutions of (13.4) of a specially simple form, viz.:

for Family I—

$$z_1 = u_1 = 0, \qquad z_2 = z_2^{\mathbf{0}}, \qquad u_2 = u_2^{\mathbf{0}} ; \qquad \ldots \ldots \quad (13.7)$$

for Families II, III—

$$z_1 = z_1^{\mathbf{0}} (\gamma e^{\sigma t})^{\lambda_0/\nu_0}, \qquad u_1 = u_1^{\mathbf{0}} (\gamma e^{\sigma t})^{-\lambda_0/\nu_0}, \qquad z_2 = z_2^{\mathbf{0}}, \; u_2 = u_2^{\mathbf{0}} ; \quad \ldots \quad (13.8)$$

but the conditions for solutions of this form will now depend on μ as well as on $(\sigma - \nu)$ and the z_k, u_k.

Cases I–V: $\lambda \neq 0$. It will be sufficient to investigate Family I. The equations (13.4) then have the equilibrium solution (13.7), provided

$$\frac{\partial \mathrm{K}}{\partial z_2} = 0, \qquad \frac{\partial \mathrm{K}}{\partial u_2} = 0. \quad \ldots \ldots \ldots \ldots \quad (13.9)$$

The argument still applies by which, if

$$\mathrm{K} \equiv (\sigma - \nu)(c_2 z_2 + d_2 u_2) + \lambda v - \tfrac{1}{2} a z_2^2 + \mu (c_2' z_2 + d_2' u_2) + c_3 (\sigma - \nu) v + c_4 z_2 v + \ldots,$$

we prove that c_2 may always be supposed non-zero, that the second of (13.9) is a consequence of the first, and that in this condition, viz.,

$$c_2 (\sigma - \nu) - a z_2 + c_2' \mu + \ldots = 0, \quad \ldots \ldots \ldots \quad (13.10)$$

we lose no generality by putting $u_2 = 0$. If $a \neq 0$ (the general case) we may solve (13.10) for z_2 as a power series in $(\sigma - \nu)$, μ; if $a = 0$ we may solve for $(\sigma - \nu)$ in powers of z_2, μ. Substituting this solution in (13.3), together with $z_1 = u_1 = u_2 = 0$, we obtain a formally periodic solution of (13.1), the x_k, y_k appearing in general as power series in $(\sigma - \nu)$, μ, in which the coefficients are Laurent series in $\gamma e^{\sigma t}$. An easy extension of the method of § 10 proves these series absolutely convergent when $|\sigma - \nu|$, $|\mu|$ are sufficiently small and $|\gamma e^{\sigma t}|$ lies between suitable limits. Thus *Family I varies continuously with* μ.

The non-zero exponents of such a solution have (if $\lambda \neq \frac{1}{2}\nu$*) the values $\pm m$, where

$$m = (\partial K/\partial v)_{z_1 = u_1 = u_2 = 0,\, z_2 = z_2^0},$$

and is so in general a power series in $(\sigma - \nu)$, μ which may be proved convergent by an extension of the method of § 11. Now the work of §§ 6, 7, has shown the importance for any periodic solution of the *characteristic ratio* (exponent)/(parameter) as affecting the nature of the results obtained for periodic solutions in its neighbourhood. We enquire therefore whether for small values of $|\mu|$ Family I will still contain a solution having the same characteristic ratio λ/ν as the generating solution (13.2). For this the condition is $m/\lambda = \sigma/\nu$, *i.e.*,

$$\left(\frac{\partial K}{\partial v}\right)_0 - \lambda = \frac{(\sigma - \nu)\,\lambda}{\nu}, \quad \ldots \ldots \quad (13.11)$$

i.e.,

$$\left(c_3 - \frac{\lambda}{\nu}\right)(\sigma - \nu) + c_4 z_2 + \ldots = 0.$$

This and (13.10) are *uniquely* soluble for $(\sigma - \nu)$, z_2 in powers of μ, provided

$$c_2 c_4 + a\,(c_3 - \lambda/\nu) \neq 0, \quad \ldots \ldots \quad (13.12)$$

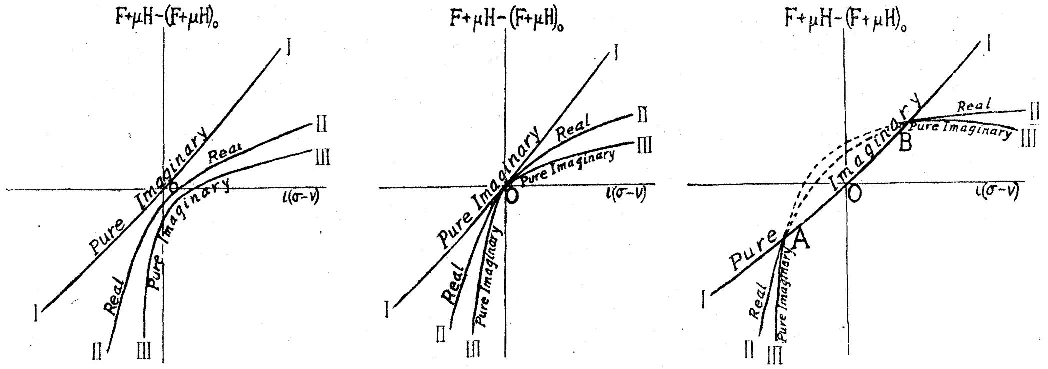
which is true in general; but if $c_2 c_4 + a\,(c_3 - \lambda/\nu) = 0$ there is in general a multiple solution in fractional powers of μ. The only case in which there can be no solution is that in which for $\mu = 0$ all the solutions of Family I have the same characteristic ratio. It is easily seen that when $c_2 c_4 + a\,(c_3 - \lambda/\nu) = 0$ the characteristic ratio of the solutions of Family I for $\mu = 0$ is *stationary* in value at the generating solution (13.2); this solution is therefore to be regarded as multiple, the multiplicity being, of course, equal to the number of solutions, having the same characteristic ratio, into which it splits when $\mu \neq 0$. Now as generating solution (13.2) when $\mu = 0$, we can take any one of the periodic solutions investigated in §§ 6, 7, 9 for which the characteristic ratio is not zero, this solution being regarded as multiple in the circumstances just explained. We see that *in general, to each periodic solution of non-zero characteristic ratio corresponds, for* $|\mu|$ *sufficiently small, a unique periodic solution having the same characteristic ratio; as* μ *varies each solution changes continuously* (*i.e., the* x_k, y_k *are for it continuous functions of* μ, t) *so as to have always the same characteristic ratio, and its parameter* σ *varies continuously with* μ. *The only case in which, as* μ *varies from the value* 0, *such a periodic solution cannot change continuously into a solution having the same characteristic ratio is that in which for* $\mu = 0$ *the solution in question belongs to a family along which the characteristic ratio is invariable; but here it is still true that the family changes continuously with* μ (the solutions composing the family remaining, of course, always periodic).

When $F + \mu H$ is a real function of μ and the x_k, y_k, and the generating solution (13.2)

* When $\lambda = \frac{1}{2}\nu$ the investigation is similar.

is real and of real period, we may show, as in § 6, p. 162, and § 7, p. 174, that the left-hand sides of the conditions (13.10), (13.11) are power series in $\iota(\sigma - \nu)$, z_2, μ with real coefficients ; hence, when the solution for $\iota(\sigma - \nu)$, z_2 in terms of μ is unique it is real, and when it is multiple complex solutions occur in conjugate pairs. Thus, *as μ varies a periodic solution of given non-zero characteristic ratio can change from reality to unreality only by uniting with another real periodic solution of the same characteristic ratio.**

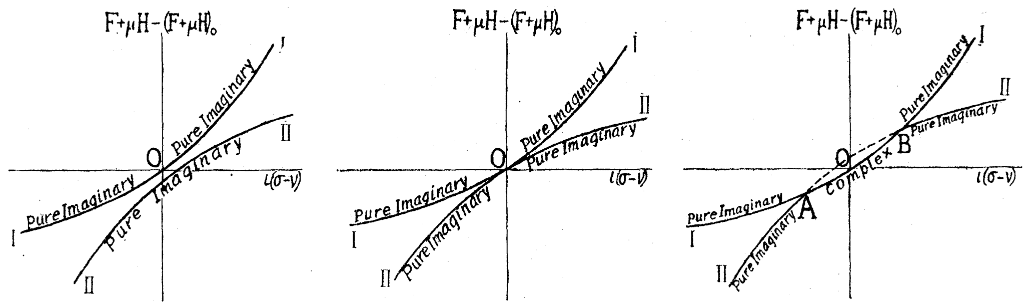
It is of interest to give figures analogous to those at the end of § 7 illustrating the relations between the real solutions of Families I, II, III, when for $\mu = 0$ there is a *double* periodic solution having its characteristic ratio commensurable ; this is an exceptional case which was not treated in § 7, where we took only the general case in which (13.12) is satisfied. If we still suppose $c_6 \neq 0$, $c_7 \neq 0$, we obtain figures analogous to fig. 1, of which the following are typical† :



$\mu < 0$. $\mu = 0$. $\mu > 0$.

$\nu_0 > 4$, $a \neq 0$, $c_6 \neq 0$, $c_7 \neq 0$, $ac'_3 + c_2c_4 = 0$.

FIG. 8.



$\mu < 0$. $\mu = 0$. $\mu > 0$.

$\nu_0 = 2$, $a \neq 0$, $c_6 \neq 0$, $ac_9 + c_2c_{14} = 0$.

FIG. 9.

* In comparing these theorems with those in "Méth Nouv.," Chap. xxxi, it must be borne in mind that POINCARÉ appears always to speak as though to any periodic solution for $\mu = 0$ corresponds for any non-zero value of μ a *unique* solution which is its analytic continuation. This is, however, not the case, since for any μ there is a continuous family of periodic solutions. To make the correspondence precise we must specify, *e.g.*, as in the text, a particular member of the family.

† As in preceding figures, along each curve is indicated the nature of the non-zero exponents of the solutions represented by its points.

We see that for $\mu = 0$ there is in Family I a double solution of characteristic ratio λ_0/ν_0 (represented by O); for $\mu > 0$ there are two real solutions of this characteristic ratio (represented by A, B), and for $\mu < 0$ no such real solution. As μ increases through negative values the (real) Families II, III gradually approach Family I: they have a double solution (O) in common with that Family for $\mu = 0$, and as μ increases through positive values they have two gradually separating solutions (A, B) in common.

We may now fill in a gap that was left in § 7, where it was stated that certain coefficients in the series K are in general non-zero. Suppose the equations (5.1) whose Hamiltonian function is F (x_k, y_k) possess a periodic solution of parameter ν and characteristic ratio λ/ν; then the equations (13.1) whose Hamiltonian function is $F + \mu H$ possess a periodic solution of parameter ν' and the same characteristic ratio, and this solution and its parameter ν' vary continuously with μ. Take this solution as generating solution, and by the method of § 5 transform the equations (13.1) into the form (5.21); then the coefficients in the new Hamiltonian function K are power series in μ which it is not difficult to prove convergent,* so these coefficients vary continuously with μ. If we suppose F defined by the coefficients in its development about some point (x_k, y_k), it follows then that $c_2, c_3, \ldots$ are analytic functions of these coefficients. Now we have given in § 8 a particular case in which c_6, c_7, $ac'_3 + c_2c_4$, $c^2_4 + ac_{10}$ are non-zero; it follows that they must be non-zero *in general* for any Hamiltonian function F. This point is of prime importance, since on it rests our conclusion that *for Hamiltonian systems in general* the periodic solutions are in general "singular" and not "ordinary."

CASE VI: $\lambda = 0$.—To obtain periodic solutions of (13.1) we now search for equilibrium solutions of (13.4), where K is a power series in μ and the z_k, u_k, not involving t, with coefficients linear in $(\sigma - \nu)$, say

$$K \equiv (\sigma - \nu)(c_1z_1 + d_1u_1 + c_2z_2 + d_2u_2) + az_2u_1 + \tfrac{1}{2}bz_1^2 + cz_1z_2 + \tfrac{1}{2}dz_2^2 + \mu(c'_1z_1 + d'_1u_1 + c'_2z_2 + d'_2u_2) + \ldots \quad (13.13)$$

The dependence of K on μ makes no difference to the argument of § 9, whereby we show that we may always suppose $c_2 \neq 0$, that the equation $\partial K/\partial u_2 = 0$ is a consequence of

$$\frac{\partial K}{\partial z_1} = \frac{\partial K}{\partial u_1} = \frac{\partial K}{\partial z_2} = 0, \quad (13.14)$$

and that in these equations we lose no generality by putting $u_2 = 0$. Explicitly (13.14) are

$$\left.\begin{aligned} c_1(\sigma - \nu) + bz_1 + cz_2 + c'_1\mu + \ldots &= 0 \\ d_1(\sigma - \nu) + az_2 + d'_1\mu + \ldots &= 0 \\ c_2(\sigma - \nu) + au_1 + cz_1 + dz_2 + c'_2\mu + \ldots &= 0 \end{aligned}\right\}.$$

* The essential points are that the series involved in the linear contact transformation (5.11) are convergent, and that the coefficient of a term of any specified order in K depends only on the coefficients of terms up to this order in G.

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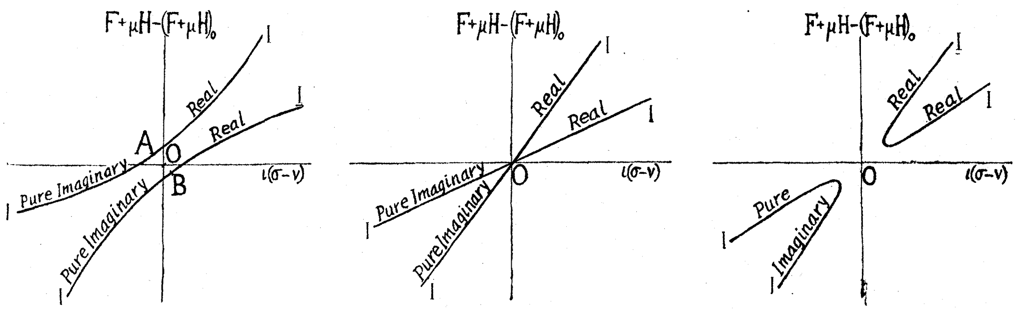
If either of the quantities a^2b, $acc_1 + (bd - c^2)\, d_1 - abc_2$ is non-zero, we may solve for three of $(\sigma - \nu)$, z_1, u_1, z_2 as power series in the fourth and μ, so there is for (13.1) a family of periodic solutions which varies continuously with μ.

Similarly if for $\mu = 0$ the equations (13.14) are independent, so that there is a solution of finite multiplicity (p, say) for, say, z_1, u_1, z_2 in terms of $(\sigma - \nu)$, then when $\mu \neq 0$ the equations are still independent, and have a solution of multiplicity p for z_1, u_1, z_2 in terms of $(\sigma - \nu)$, μ. There are, then, for (13.1) p families of periodic solutions which vary continuously with μ, and have the generating solution as a common member when $\mu = 0$. For instance, if $a = 0$, $bd - c^2 \neq 0$, the relations between the real periodic solutions in the neighbourhood of the generating solution are in general represented by figures such as the following (*cf.* fig. 7):



$\mu < 0$. $\mu = 0$. $\mu > 0$.

$\lambda = 0, \quad a = 0, \quad bd - c^2 \neq 0.$

FIG. 10.

Here for $\mu < 0$ we have two real periodic solutions having all their exponents zero (represented by A, B); for $\mu = 0$ a double solution (O) of this nature, and for $\mu > 0$ no real such solution.

It may happen that for $\mu = 0$ the equations (13.14) are not independent.* If this is so there is in general no solution for any three of $(\sigma - \nu)$, z_1, u_1, z_2 in terms of the fourth and of μ, which vanishes with these quantities. We then have in the neighbourhood of the generating solution a *double infinity* of periodic solutions when $\mu = 0$, and *no* such solution when $\mu \neq 0$. For instance, if $a = 0$, $bd - c^2 \neq 0$ we may solve the first and third of (13.14) for z_1, z_2 in powers of u_1, $(\sigma - \nu)$, μ, and on substitution in the second it becomes a power series in u_1, $(\sigma - \nu)$, μ. If the terms independent of μ vanish identically the equations (13.1) have for $\mu = 0$ a double infinity of periodic solutions (depending on the two arbitrary parameters u_1, $(\sigma - \nu)$). When $\mu \neq 0$ we may cancel a factor μ or some power of μ, and the resulting condition, say

$$A + Bu_1 + C(\sigma - \nu) + D\mu + \ldots = 0, \quad \ldots \ldots \quad (13.15)$$

is not soluble for u_1 in powers of $(\sigma - \nu)$, μ or for $(\sigma - \nu)$ in powers of u_1, μ unless $A = 0$, *which is not, in general, the case.*† This state of affairs is of importance because it arises

* Note that in Cases I–V the corresponding equations $z_1 = u_1 = \partial K/\partial z_2 = 0$ are *always* independent.

† The justification of this statement is by an argument similar to that employed above concerning c_6, c_7, ... See also the next paragraph but one.

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when for $\mu = 0$ the equations (13.1) form a "soluble" system of which the generating solution (13.2) is an "ordinary" periodic solution (§ 2 above). To pursue the matter further suppose that when $\mu = 0$ the system (13.1) is "soluble," and consider a definite singly-infinite family $\mathfrak{F}$ of ordinary periodic solutions, all having the same period, $\mathfrak{F}$ depending continuously on an arbitrary parameter α. Taking any solution of this family as generating solution we may form the conditions (13.14), and thence the condition (13.15), and here the coefficients A, B, . . . will depend continuously on α. The conditions that (13.15) may have a solution for u_1 in terms of $(\sigma - \nu)$, μ which vanishes with these quantities (or for $(\sigma - \nu)$ in terms of u_1, μ) are (i) $A = 0$, (ii) the left-hand side must not vanish when $\mu = 0$, this being so, for example, if $B \neq 0$ or $C \neq 0$. In general, A will not vanish identically as regards α, but it may be possible to give α a discrete set of values $\alpha_1, \alpha_2, \ldots$ for any one of which $A = 0$. If for $\alpha = \alpha_1$ the left-hand side of (13.15) does not vanish with μ, we obtain a solution (or a finite number of solutions) of the required nature for u_1 in terms of $(\sigma - \nu)$, μ or for $(\sigma - \nu)$ in terms of u_1, μ; that solution of the family $\mathfrak{F}$ for which $\alpha = \alpha_1$ leads then to a singly-infinite family (or to a finite number of such families) of periodic solutions which varies continuously with μ. In this case, then, *the doubly-infinite family of periodic solutions which exists when* $\mu = 0$ *leads when* $\mu \neq 0$ *to a finite number of singly-infinite families of periodic solutions.*

We may note how this conclusion fits in with a remark made at the end of § 7 (i) (p. 170 above). We there saw that in special cases there may branch from Family I at the generating solution a doubly-infinite family of periodic solutions instead of a finite number of singly-infinite families: the cases in question are, of course, those in which the system (5.1) is "soluble." There is, then, an obvious connection between the statement just italicised and the fact that in § 7 the *general* case is that in which there branch from the generating solution two singly-infinite families of periodic solutions and not a doubly-infinite family. We may sum up by saying that *for an arbitrary Hamiltonian system the periodic solutions are in general "singular."*

It is easily shown that *the theorems italicised on p. 212 above are true of a periodic solution of zero characteristic ratio which belongs to a singly- but not a doubly-infinite family of periodic solutions.*

Summary and Synthesis.

§ 14. *On the Totality of all Regular* Periodic Solutions possessed by a given Hamiltonian System.*

The preceding investigations have all been purely "local" in character; starting from a periodic solution $\mathfrak{S}$ we have shown how to find, for a *sufficiently small neighbourhood of* $\mathfrak{S}$, families of periodic solutions to which $\mathfrak{S}$ belongs. Our object is now as far as possible to synthesise the infinite number of scraps of "local" knowledge thus obtainable so as

* For the explanation of this term see § 10, pp. 185, 186, above.

to gain some idea of the mutual relations of *all* periodic solutions possessed by a given Hamiltonian system of equations. This appears to be a problem of great complexity, and certain parts of what follows are distinctly speculative in character. The essence of the complication is this: that whereas the investigation of §§ 6, 7 or 9 furnishes periodic solutions in the neighbourhood of $\mathfrak{S}$, it certainly does not furnish *all* the periodic solutions in this neighbourhood. In fact, it appears that the problem of finding all the periodic solutions in any neighbourhood of $\mathfrak{S}$, however small, may be of the same order of complication as the similar problem in which we do not thus restrict ourselves to a specific neighbourhood.

(i) We start by summing up the chief results of the preceding work.

THEOREM I.—*A fourth order Hamiltonian system*

$$\frac{dx_k}{dt} = \frac{\partial \mathrm{F}}{\partial y_k}, \qquad \frac{dy_k}{dt} = -\frac{\partial \mathrm{F}}{\partial x_k}, \qquad \mathrm{F} \equiv \mathrm{F}(x_k, y_k), \quad . \quad . \quad . \quad . \quad . \quad (14.1)$$

possesses regular periodic solutions if there exists a point $(x_k{}^0, y_k{}^0)$ *at which the derivatives* $\partial \mathrm{F}/\partial x_k$, $\partial \mathrm{F}/\partial y_k$ *all vanish, and at which the analytic function* F *is regular* (§ 12); *it also possesses such solutions if* F *differs sufficiently little from a Hamiltonian function* F_0 *for which the equations are known to possess a regular periodic solution* (§ 13). We do not enquire further into the necessary conditions that F must satisfy in order that (14.1) may possess periodic solutions. (For references see the end of § 4 above.)

THEOREM II.—*If the system* (14.1) *possesses a regular periodic solution* $\mathfrak{S}$ (*not an equilibrium solution*), *it possesses also a continuous singly-infinite family* $\mathfrak{F}$ (*of which* $\mathfrak{S}$ *is a member*) *of such solutions, for which the parameter* σ *and the characteristic ratio* R *vary continuously along the family.*

Analytically the family $\mathfrak{F}$ is specifiable in the form

$$x_k = \phi_k(p, e^{\sigma t}), \qquad y_k = \psi_k(p, e^{\sigma t}), \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (14.2)$$

where the ϕ_k, ψ_k, are absolutely convergent series in positive powers of p and positive and negative powers of $e^{\sigma t}$, and σ is an absolutely convergent series of positive powers of p; the solution $\mathfrak{S}$ is given by $p = 0$. In general we can express p (and therefore also the ϕ_k, ψ_k) as a power series in $(\sigma - \nu)$, where ν is the parameter of $\mathfrak{S}$; the exceptional case is that in which $dp/d\sigma$ vanishes for $\sigma = \nu$, and p and the ϕ_k, ψ_k are then in general expressible in powers of $(\sigma - \nu)^{1/n}$, where n is an integer > 1 (§§ 6, 7 (i), 9, 10, 11).

THEOREM IIA.—*When* F *is a real function of the* x_k, y_k *and the solution* $\mathfrak{S}$ *is real and of real period, those solutions of the family* $\mathfrak{F}$ *are real for which the period is real. For all such real solutions near* $\mathfrak{S}$ *the characteristic ratio* R *is real or unreal, according as the characteristic ratio* R_0 *of* $\mathfrak{S}$ *is real or unreal; except that when* $R_0 = 0$, R *in general changes from real to pure-imaginary as we pass through* $\mathfrak{S}$ *along the* (*real*) *family, and when* $R_0 = \frac{1}{2}$, R *similarly changes from real to complex.* (§§ 6, 7 (iii), 9.)

THEOREM III.—*If for the solution* $\mathfrak{S}$ *the characteristic ratio has a real non-zero commensurable value, viz., the ratio of two integers* λ_0, ν_0 *mutually prime, there branch from the family* $\mathfrak{F}$ *at* $\mathfrak{S}$ *a finite number (in general) of other singly-infinite families of regular periodic solutions, of which the parameter tends continuously in the neighbourhood of* $\mathfrak{S}$ *to the value of* ν/ν_0 *and the characteristic ratio to the value* 0. *In general there are two such families,* $\mathfrak{F}_1$, $\mathfrak{F}_2$. Their analytical specification is of the form

$$x_k = \phi_k\{(\sigma - \nu)^{\frac{1}{2}}, e^{\sigma t/\nu_0}\}, \qquad y_k = \psi_k\{(\sigma - \nu)^{\frac{1}{2}}, e^{\sigma t/\nu_0}\}, \quad . \quad . \quad . \quad . \quad (14.3)$$

where the ϕ_k, ψ_k are absolutely convergent series of positive powers of $(\sigma - \nu)^{\frac{1}{2}}$ and positive and negative powers of $e^{\sigma t/\nu_0}$. When $\nu_0 = 2$ or $\nu_0 = 3$ there is in general only one such family. (§ 7 (i), **10**, **11**.)

THEOREM IIIA.—*When* F *is a real function of the* x_k, y_k *and the solution* $\mathfrak{S}$ *is real and of real period, both families* $\mathfrak{F}_1$, $\mathfrak{F}_2$ *possess real solutions of real period, forming continuous real families* $\mathfrak{F}'_1$, $\mathfrak{F}'_2$; *in general these real families are limited by the solution* $\mathfrak{S}$, *i.e., they extend on one side only of* $\mathfrak{S}$. *The characteristic ratio is in general real for the solutions composing* $\mathfrak{F}'_1$ *and pure-imaginary for those composing* $\mathfrak{F}'_2$. (§ 7 (iii).)

THEOREM IV.—*In the neighbourhood of a regular equilibrium solution* $\mathfrak{E}$ *there are, in general, two or more families of regular periodic solutions having* $\mathfrak{E}$ *as a limiting member.* Analytically, such a family is in general of the form

$$x_k = \phi_k\{(\sigma - \nu)^{\frac{1}{2}}, e^{\sigma t}\}, \qquad y_k = \psi_k\{(\sigma - \nu)^{\frac{1}{2}}, e^{\sigma t}\},$$

where the ϕ_k, ψ_k are series of the same nature as those in (14.2), (14.3) and the equilibrium solution $\mathfrak{E}$ is given by $\sigma = \nu$. (§ 12.)

THEOREM V.—*When in the equations* (14.1) F *is an analytic function of an arbitrary parameter* μ, *each family of periodic solutions varies continuously with* μ, *and in general each periodic solution can be continuously correlated with one having the same characteristic ratio.* (§ 13.)

THEOREM VA.—*When* F *is a real function of* μ *and the* x_k, y_k, *a periodic solution of given characteristic ratio can, in varying with* μ, *change from reality to unreality only by uniting with another of the same characteristic ratio.* (§ 13.)

THEOREM VI.—*The occurrence of a doubly-infinite family of regular periodic solutions is to be regarded as exceptional* (§ 13).

Suppose that $\mathfrak{S}$ is a real regular periodic solution of real period and real characteristic ratio ; then, from these results, it is easily seen that there must be in general an infinity of real families of regular periodic solutions in the neighbourhood of $\mathfrak{S}$. We have first of all the family $\mathfrak{F}$ (Family I of §§ 6, 7, 9) to which $\mathfrak{S}$ belongs ; from all "rational points" of this (*i.e.*, solutions whose characteristic ratio is rational) there branch two real families, $\mathfrak{F}_1$, $\mathfrak{F}_2$, and for each $\mathfrak{F}_1$ the characteristic ratio is real and continuous ; from each "rational point" of each $\mathfrak{F}_1$ there branch two new families, and so on, and since for any family rational points are everywhere dense there are infinitely many families traversing an arbitrarily small neighbourhood of $\mathfrak{S}$. Our problem is to unravel the mutual relations of these families.

(ii) *The Region of Convergence of a Family of Regular Periodic Solutions.*—The investigations of §§ 5–11 furnish the equations of such a family in general in the form

$$x_k = \phi_k(\sigma, e^{\sigma t}), \qquad y_k = \psi_k(\sigma, e^{\sigma t}), \quad \ldots \ldots \ldots \quad (14.4)$$

where the ϕ_k, ψ_k are Laurent series in $e^{\sigma t}$ with coefficients which are convergent power series in $(\sigma - \nu)$ or in $(\sigma - \nu)^{1/n}$. These coefficients are thus analytic functions of σ, and over any range of σ for which their analytic continuation is possible (14.4) must remain a periodic solution of the equations. Moreover, if at any singularity $\sigma = \sigma_0$ of one or more of these coefficients the functions ϕ_k, ψ_k remain determinate Laurent series in $e^{\sigma_0 t}$, (14.4) will remain a periodic solution for $\sigma = \sigma_0$ and may be called a *singular* member of the family (14.4). Now the preceding work has shown under what conditions a regular periodic solution $\mathfrak{S}$ is a singular member of a family $\mathfrak{F}$. Suppose that (14.4) represents the family $\mathfrak{F}$ and that $\mathfrak{S}$ corresponds to $\sigma = \sigma_0$; then we have seen in (i) above that the ϕ_k, ψ_k are developable in powers of $(\sigma - \sigma_0)$ unless

(*a*) the parameter σ (or period T) for the family $\mathfrak{F}$ is stationary in value at $\sigma = \sigma_0$; or

(*b*) the family $\mathfrak{F}$ degenerates for $\sigma = \sigma_0$ into a solution having the parameter $n\sigma$ where n is an integer greater than 1, so that $\mathfrak{F}$ branches from another family at $\sigma = \sigma_0$; or

(*c*) the family $\mathfrak{F}$ degenerates for $\sigma = \sigma_0$ into an equilibrium solution; or

(*d*) the solution for which $\sigma = \sigma_0$ is not a regular periodic solution. *The singular solutions of a family are those for which one of the conditions* (*a*), (*b*), (*c*), (*d*) *is satisfied, and the circle of convergence* C *in the* σ*-plane of the development of the functions* ϕ_k, ψ_k *about the point* $\sigma = \nu$ *goes through the nearest point* $\sigma = \sigma_0$ *for which the solution* (14.4) *is singular.*

It is evident that (*c*) is not a general case; and there are reasons (which do not amount to a proof and will not be given here) to suppose also that (*a*) and (*d*) are not. *It is reasonable to assume that in general the "singular" solutions of a family are "degenerate" members of it in the sense of* (*b*) *above.*

Now the range of convergence found in § 10 for the developments of Families II, III, in powers of $(\sigma - \nu)^{\frac{1}{2}}$ is governed by the magnitude of the integer ν_0, and though we have found there only a lower limit to the actual range of convergence, which is, say, $|\sigma - \nu| \leq s_0$, we may plausibly suppose that s_0 tends to zero as $\nu_0 \to \infty$. Combining with the assumption just made we see: *it is reasonable to assume that the smaller the parameters of the solutions of a family, the nearer together will occur "degenerate" members of it.*

(iii) *Classification and Geometrical Representation of Real Periodic Solutions.*—We take the well-known geometrical representation in which the x_k, y_k are rectangular co-ordinates in four-dimensional space; any real solution is then represented by a curve or *trajectory* in this space, and for a periodic solution the corresponding trajectory is a closed curve.

To gain some idea of the nature of the mutual relations between the infinitely many families of periodic solutions which have been seen to exist, it is advantageous to suppose that the system (14.1) is such that through any point of the space there passes an analytic three-dimensional "surface" $\mathfrak{V}$, closed or infinite, which is touched by no trajectory.* We call $\mathfrak{V}$ a *surface of section*.

The existence of such a surface of section is closely connected with that of two-dimensional surfaces of section as studied by POINCARÉ and BIRKHOFF (*loc. cit.*). We do not enquire here what restriction, if any, is imposed on the system (14.1) by making this hypothesis. As a simple example, take

$$F \equiv f(y_1/x_1, x_2, y_2) \cdot e^{-\frac{1}{2}(x_1^2+y_1^2)},$$

where f is one-signed for all real x_1, y_1, x_2, y_2; then it will be found that

$$\frac{d(\tan^{-1} y/x)}{dt} = F = \text{constant},$$

so any surface of the family y/x = constant is a surface of section. This example illustrates the fact that the existence of a surface of section does not imply that the system is "soluble."

Any periodic trajectory cuts $\mathfrak{V}$ in a finite number of points, and from reasons of continuity combined with the characteristic property of $\mathfrak{V}$ it is apparent that *for trajectories which correspond to periodic solutions of any one family* $\mathfrak{F}$ *the number of such intersections is the same.* We define the *order* of a periodic solution as the number of points in which the corresponding trajectory cuts $\mathfrak{V}$. Along any family $\mathfrak{F}$, then, the order n is invariable (and, of course, integral), and the family is represented within $\mathfrak{V}$ by n analytic curves $\mathfrak{L}_1, \ldots \mathfrak{L}_n$, in which the corresponding family of trajectories cuts $\mathfrak{V}$.

Now let $\mathfrak{F}_1$ be a real family of periodic solutions which branches from $\mathfrak{F}$ at a solution $\mathfrak{S}$, having the period T and characteristic ratio λ_0/ν_0. Then for $\mathfrak{F}_1$ the period tends to the value ν_0T in the neighbourhood of $\mathfrak{S}$, so clearly $\mathfrak{F}_1$ is of order $n\nu_0$, and is represented within $\mathfrak{V}$ by $n\nu_0$ analytic curves which radiate in sets of ν_0 from the n points on $\mathfrak{L}_1, \ldots \mathfrak{L}_n$ respectively which represent $\mathfrak{S}$. Thus, *provided there exists a surface of section, the whole aggregate of families of periodic solutions is represented by a connected aggregate* $\mathfrak{A}$ *of curves in a three-dimensional space.* The relations between any group of families are most easily pictured by reference to $\mathfrak{A}$.

This geometrical representation forms a natural extension of our previous one, which used the branching of curves in a plane (figs. 1–10), with the advantage that by going from two dimensions to three we are able to represent *all* families on the one figure.

Now suppose, in accordance with the supposition at the end of (ii) above, that a family $\mathfrak{F}$ of order n possesses two "degenerate" solutions $\mathfrak{S}_1$, $\mathfrak{S}_2$ of orders n_1, n_2 respectively, so that n/n_1, n/n_2 are integers > 1. Then $\mathfrak{S}_1$, $\mathfrak{S}_2$ belong to families $\mathfrak{F}_1$, $\mathfrak{F}_2$ of orders n_1, n_2 respectively, and these are connected through $\mathfrak{F}$. If another family

* $\mathfrak{V}$ is not allowed to have an "edge" of two dimensions or less, for the trajectories through the point of such an edge are counted as touching $\mathfrak{V}$.

$\mathfrak{F}'$ also has "degenerate" members in common with $\mathfrak{F}_1$, $\mathfrak{F}_2$, we shall evidently have belonging to $\mathfrak{A}$ a finite number of closed curvilinear polygons, made up of portions of the curves which represent $\mathfrak{F}$, $\mathfrak{F}'$, $\mathfrak{F}_1$, $\mathfrak{F}_2$. In figs. 8 and 9 ($\mu > 0$) we have illustrated a case in which an $\mathfrak{F}$ has two "degenerate" members in common with the same $\mathfrak{F}_1$, and examples may be given of equations for which distinct families $\mathfrak{F}_1$, $\mathfrak{F}_2$ are connected by several families, as above. We thus reach the conception that *the aggregate* $\mathfrak{A}$ *consists of a network of closed meshes*; and it is in accordance with the last sentence of (ii) above to speculate further that *the side-length of a mesh tends to zero as the order of the corresponding family tends to infinity.*

Concerning the aggregate $\mathfrak{A}$ there arise several further questions of which the answers would provide information of great interest, *e.g.*:

(*a*) What are the rules which relate the orders of successive "degenerate" members of a family of given order ?

(*b*) Given two families of known orders, what are the orders of the families which connect them, and are such families finite or infinite in number ?

(*c*) Are there points of $\mathfrak{A}$ arbitrarily near to each point of $\mathfrak{V}$?

(*d*) What is the asymptotic behaviour of the n points which represent a solution of order n when n is large, *e.g.*, do they tend to become everywhere dense in $\mathfrak{A}$?

It seems that there is little hope of making progress towards answering these questions by using, as throughout the preceding work, power series for the specification of the families. Moreover, simple illustrative examples are lacking because we are concerned here with "insoluble" and not with "soluble" systems.